String Field Theory: A Review

弦场论综述

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Abstract

摘要

As of today, there exist consistent, gauge-invariant string field theories describing all string theories: bosonic open and closed strings, open superstrings, heterotic strings, and type II strings. The construction of these theories requires algebraic ingredients, such as A_∞ and L_∞ homotopy algebras; geometric ingredients, relevant to the building of moduli spaces of Riemann surfaces and the distribution of picture changing operators; and field-theoretic ingredients, involving two-dimensional CFT's and BCFT's and Batalin-Vilkovisky quantization. Applications of string field theory include the description of non-perturbative phenomena such as tachyon condensation and classical solutions and the resolution of a number of ambiguities that bedevil the world-sheet formulation of perturbative string theory. It also allows, given a proper definition of contours

of integration for momenta, for a proof of unitarity and a clear understanding of the ultraviolet finiteness of the theory. In this chapter, we review these developments.

目前学界已存在描述所有弦理论的相容规范不变弦场理论，涵盖玻色开弦与闭弦、开超弦、杂化弦以及 II 型弦。构建这些理论需要三类要素: 代数要素，例如 A_∞ 和 L_∞ 同伦代数；几何要素，和黎曼曲面模空间构造以及图变算子分布相关；场论要素，涉及二维共形场论 CFT、边界共形场论 BCFT 与巴塔林-维尔可维斯基量子化。弦场理论可用于描述非微扰现象，比如快子凝聚与经典解，还能解决微扰弦理论世界面表述中存在的诸多歧义。在给定正确动量积分围道定义的前提下，弦场理论还可证明理论的幺正性，并清晰阐明其紫外有限性。本章我们将综述这些研究进展。

Keywords

关键词

String field theory · Properties of string field theory

弦场理论 · 弦场理论性质

Introduction

引言

The modern era of string field theory began in 1984, when Warren Siegel was able to write Lorentz-covariant, gauge-fixed, free field theories for open and closed bosonic strings [1]. His work used the previously discussed BRST first quantization of the string by Kato and Ogawa [2]. String field theories in the light-cone gauge had been formulated in the 1970s by Kaku and Kikkawa [3] based on the interacting string picture developed by Mandelstam [4]. Siegel's work was the starting point in the formulation of gauge-invariant string field theory—modern string field theory. As of 2024, gauge-invariant string field theory has been studied for 40 years.

弦场论的现代纪元始于 1984 年，当年沃伦·西格尔成功写出了洛伦兹协变、规范固定的开、闭玻色弦自由场论 [1]。他的工作用到了加藤与小川此前讨论过的弦 BRST 一阶量子化结果 [2]。光锥规范下的弦场论早在 20 世纪 70 年代就由加古和吉川基于曼德尔施塔姆提出的相互作用弦图像完成了构造 [3][4]。西格尔的工作是构造规范不变弦场论——即现代弦场论的起点。截至 2024 年，规范不变弦场论的研究已经走过了 40 个年头。

The first challenge in this subject, of course, was the formulation of the various string field theories. For bosonic strings, this includes classical open string field theory, classical and quantum closed string field theory, and open-closed string field theory-by definition, a quantum theory. For supersymmetric strings, there were separate complications with the Neveu-Schwarz (NS) and the Ramond (R) sectors. Classical open superstrings have both sectors. Heterotic strings also have both sectors, and type II strings combine these two sectors to form NS-NS, NS-R, R-NS, and R-R sectors. Much had to be learned to accomplish the construction of the various string field theories. Fully interacting, quantum bosonic string field theories were essentially complete by 1993. Superstrings brought new challenges. By 2015, however, we have had consistent gauge-invariant string field theories for all supersymmetric closed strings. The construction of open-closed superstring field theory was completed a few years later. This is not to say these formulations are final-there are a number of works with partial success in different alternative directions, but the complete theories we now have pass all known consistency checks.

该领域的首要挑战当然是构造各类弦场论。对于玻色弦来说，这包括经典开弦场论、经典与量子闭弦场论，以及定义上属于量子理论的开-闭弦场论。对于超对称弦，内沃-施瓦茨 (NS) 区和拉蒙德 (R) 区各有不同的复杂问题。经典开超弦同时包含这两个区，杂化弦也同时包含两个区，而 II 型弦结合这两个区后形成了 NS-NS、NS-R、R-NS 和 R-R 四个区。要完成各类弦场论的构造需要解决大量新问题。到 1993 年，全相互作用量子玻色弦场论已基本构建完成。超弦带来了新的挑战，但截至 2015 年，我们已经得到了所有超对称闭弦的自洽规范不变弦场论。开-闭超弦场论的构造也在数年后完成。这并不意味着现有构造就是最终形式——已有不少工作在不同替代方向取得了部分进展，但我们目前拥有的完整理论通过了所有已知的自洽性检验。

One of the explicit goals of string field theory was to provide a non-perturbative definition of string theory. While this goal still remains to be attained, there was a notable success in describing tachyon condensation in open string field theory, a non-perturbative phenomenon that had been a mystery since the tachyon instability had been noticed. There has also been impressive progress in the construction of analytic solutions in open string field theory—another non-perturbative construction. Admittedly, to date, string field theory has done little to help understand holography, black holes, and a number of dualities. [1] One may have also expected string field theory to provide a background independent definition of string theory. It has been proven that string field theory has the property of background independence, at least for backgrounds that are nearby. This property, however, is not yet manifest in the present formulations, and it seems that a significant new idea may be required to obtain manifestly background independent formulations. In the meantime, background independent structures have been identified in string field theory that could play an important role in future developments.

弦场论的明确目标之一是给出弦论的非微扰定义。虽然这一目标尚未实现，但弦场论在描述开弦的快子凝聚上已经取得了值得关注的成果——快子凝聚是早在快子不稳定性被发现时就一直困扰学界的非微扰现象。此外，在开弦场论中构造解析解这另一项非微扰构造工作也取得了令人瞩目的进展。不可否认，迄今为止弦场论对理解全息、黑洞和若干对偶几乎没有帮助。[1] 人们也原本期望弦场论能给出弦论不依赖背景的定义。已经证明，弦场论至少在邻近背景范围内具备背景不依赖性，但该性质在现有构造中尚未体现出来，要得到明显背景独立的构造似乎需要重大的新思想。与此同时，弦场论中已经识别出了背景独立结构，这类结构未来的发展中可能发挥重要作用。

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[1] In this context, one should keep in mind that the action of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory also has not been used to prove the S-duality of the theory. But this does not mean that such an action is not useful.

[1] 在此背景下，我们应当记住，即使是 $\mathcal{N} = 4$ 超对称杨-米尔斯理论的作用量，也尚未被用来证明该理论的 S 对偶性，但这并不意味着这个作用量没有用处。

String field theory work began at the time when the rules of first-quantized string theory seemed adequate to describe perturbative string theory. Subsequent work, however, demonstrated that first-quantized rules are lacking and are sometimes ambiguous. In particular, in first quantization, there is no systematic way to deal with infrared divergences. Additionally, there is no systematic way to deal with S-matrix elements for states that undergo mass or wave-function renormalization. String field theory, in fact, provides the first complete definition of string perturbation theory. This is a significant success for the theory. We will review the way in which string field theory deals with infrared divergences and mass renormalization. With the understanding that string field theory is a quantum field theory, albeit with some novel properties, it has been possible to consider in a precise way some of the usual claims about string theory. We will discuss unitarity of superstring amplitudes, reviewing the prescription for dealing with the integration contour for internal loop energies, an important recent development. Finally, string field theory puts the claim of perturbative ultraviolet finiteness of string theory in a solid footing. This is important, since the original reason for considering string theory was the hope that it would be a finite theory of quantum gravity.

弦场论研究起步的时候，一阶量子化弦论的规则似乎足以描述微扰弦论。但后续研究表明，一阶量子化规则存在不足，有时还会带来歧义。具体来说，一阶量子化没有处理红外发散的系统方法，也没有系统方法处理经历质量或波函数重整化的态的 S 矩阵元。事实上，弦场论给出了弦论微扰论的第一个完整定义，这是该理论一项重大成果。我们会回顾弦场论处理红外发散和质量重整化的方法。认识到弦场论是一种量子场论 (尽管具备一些全新性质) 后，我们才能精确考察弦论的一些常见论断。我们会讨论超弦振幅的幺正性，回顾处理内部圈能量积分围道的方案——这是近年的一项重要进展。最后，弦场论为弦论微扰紫外有限性的论断打下了坚实基础。这点十分重要，因为最初研究弦论的动因就是希望它能成为一个有限的量子引力理论。

Investigations into string field theory have also had an impact on mathematics. The remarkable efficiency of Batalin-Vilkovisky (BV) quantization in dealing with (most versions of) string field theory has been extended to a discovery of BV algebras at the level of Riemann surfaces, which are the counterpart of the similar structures that exist on the state space of conformal field theories. Another subject is homotopy algebras. The cubic bosonic open string field theory [5] is based on an associative vertex, but this only works for the classical theory, and no cubic version seems to exist for open superstrings. The general framework for open string theory is that of homotopy-associative A_∞ algebras, where the cubic vertex fails to associate and that failure is repaired by higher vertices. Closed string field theory, at the classical level, provided the first explicit discussion of L_∞ algebras and their defining identities [6]. In this case, a Jacobi-like identity would guarantee a cubic gauge-invariant closed string field theory, but no such theory exists. For open-closed bosonic string field theory, constructed in [7-9], a new structure arises from the interplay between A_∞ and L_∞ structures. Homotopy algebras play an important role in some recent developments, helping, for example, construct effective actions via homotopy transfer. Finally, string field theory has helped the understanding of moduli spaces of Riemann surfaces, since the Feynman rules of the theory must build such spaces. Canonical choices of string vertices (for open and closed strings) arise via minimal area problems, which tie the theory of quadratic differentials on Riemann surfaces, and have contributed to the understanding of systolic geometry. String vertices also arise canonically in hyperbolic geometry, where they have a particularly simple definition.

对弦场论的研究也对数学产生了影响。巴塔林-维尔可夫斯基 (BV) 量子化处理 (大多数版本的) 弦场论时表现出非凡的有效性，这进一步推动人们发现了黎曼曲面层面的 BV 代数，它们是共形场论态空间上同类结构的对应。另一个研究方向是同伦代数。三次方玻色开弦场论 [5] 基于结合顶点，但这仅适用于经典理论，开超弦似乎不存在三次方版本。开弦理论的一般框架是同伦结合 A_∞ 代数: 三次方顶点不满足结合性，这种不结合性由更高阶顶点弥补。闭弦场论在经典层面给出了 L_∞ 代数及其定义恒等式的第一次明确讨论 [6]。在这种情况下，类雅可比恒等式可以保证三次方规范不变的闭弦场论，但并不存在这样的理论。对于文献 [7-9] 中构造的开-闭玻色弦场论，A_∞ 结构与 L_∞ 结构的相互作用衍生出了一种新结构。同伦代数在近年的若干发展中发挥了重要作用，例如帮助通过同伦传递构造有效作用量。最后，由于弦场论的费曼规则必须构造出这类空间，弦场论帮助人们加深了对黎曼曲面模空间的理解。通过极小面积问题可以得到弦顶点 (开弦和闭弦) 的正则选择，该问题关联了黎曼曲面上二次微分的理论，还推动了对收缩几何的理解。弦顶点也会在双曲几何中正则出现，在那里它有格外简洁的定义。

A full book would be needed to give a self-contained overview of string field theory, and in fact, an instructive introductory book has been written by H. Erbin [10]. Additionally, there are a number of reviews on various aspects of string field theory [11-18]. In this work, we have tried to focus on key ideas while dealing briefly with matters for which a number of other references exist. This review aims to be readable by graduate students, post-docs, faculty, and physicists that have had exposure to string theory (bosonic strings and superstrings) and to two-dimensional conformal field theory.

要给出弦场论的自洽概述需要一整本书，事实上 H. Erbin 已经撰写了一本有启发性的入门书 [10]。此外，还有大量关于弦场论各方面的综述文献 [11-18]。在本文中，我们聚焦核心思想，对已有大量其他文献讨论的内容仅做简要处理。本综述面向接触过弦理论 (玻色弦和超弦) 以及二维共形场论的研究生、博士后、科研人员和物理学家，力求可读性。

In section "Glossary of Symbols" we give a glossary of the various symbols used in this review. This review begins in section "Background Material" with background information on CFT, world-sheet fields, and the construction of string amplitudes in first quantization, both for bosonic strings and superstrings. We also sketch the basic setup of the Batalin-Vilkovisky formalism. This section reviews key facts that are needed throughout. In section "Bosonic and Superstring Field Theories", we attempt to give the main story of string field theory, with some of the explanation and justification postponed to later sections. So we briefly go over the full set of complete string field theories, all the bosonic ones, and all the superstrings. This includes the use of an auxiliary string field that allows for the construction of consistent string field theories for the heterotic and type II theories [19,20]. This auxiliary field turns out to represent free propagating degrees of freedom; they are needed to write the action but have no effect on the S-matrix of the interacting string fields. An auxiliary open string field is also used for the open string sector in the construction of the open-closed superstring theory given in [21]. This section concludes with very explicit calculations aimed to make some of the discussion more concrete and intuitive. We compute in detail the quadratic part of the action for the massless sector of open bosonic strings and for the massless sector of closed bosonic strings.

我们在“符号术语表”一节给出了本综述所用各类符号的释义。本综述从“背景材料”一节开始，介绍 CFT、世界面场以及一次量子化中玻色弦和超弦弦振幅构造的背景信息，也概述了巴塔林-维尔可夫斯基形式的基本框架。本节回顾了全文所需的核心结论。在“玻色弦场论与超弦场论”一节，我们梳理了弦场论的主要发展脉络，部分解释和论证推迟到后续章节。因此我们简要梳理了所有完整的弦场论，包括全部玻色弦理论和全部超弦理论。这其中包含了辅助弦场的使用: 借助辅助弦场可以构造杂化弦和 II 型理论的自洽弦场论 [19,20]。这种辅助场被证明描述的是自由传播自由度；它们是写出作用量所必需的，但不影响相互作用弦场的 S 矩阵。文献 [21] 构造开-闭超弦理论时，也在开弦部分使用了辅助开弦场。本节最后给出了非常明确的计算，让部分讨论更具体直观。我们详细计算了开玻色弦无质量 sector 和闭玻色弦无质量 sector 作用量的二次项部分。

In section ”Properties of String Field Theory”, we discuss some key facts and properties of string field theory. These include the construction of string amplitudes from string field theory, with particular attention to the propagators and their role in building forms on the moduli spaces of Riemann surfaces. We give a detailed discussion of the normalization of forms for integration over moduli spaces of Riemann surfaces, with particular attention to the case of open-closed string field theory, which is somewhat delicate [22]). We then discuss one-particle irreducible effective actions and Wilsonian actions. We review how different constructions of the moduli spaces of Riemann surfaces lead to string field theories related by field redefinitions. This is followed by a brief discussion of the proof of background independence of string field theory and the dilaton theorem.

在“弦场论的性质”一节，我们讨论弦场论的若干核心结论与性质，包括如何从弦场论构造弦振幅，尤其关注传播子及其在构造黎曼曲面模空间上形式的作用。我们详细讨论了黎曼曲面模空间积分所用形式的归一化，尤其关注开-闭弦场论的情况，这个问题相当微妙 [22]。随后我们讨论了单粒子不可约有效作用量和威尔逊作用量，我们回顾了黎曼曲面模空间的不同构造如何引出通过场重定义关联的弦场论。之后简要讨论了弦场论背景独立性的证明和伸缩子定理。

While sections ”Bosonic and Superstring Field Theories” and ”Properties of String Field Theory” review the practical approach to string theory, sections ”Algebraic Structures Underlying String Field Theories” and ”String Vertices” go into some depth on the basic mathematical tools needed to write the string field theories. The first, section ”Algebraic Structures Underlying String Field Theories”, goes over the A_∞ construction of open string theory. For closed strings, at the classical level, we develop the structure of L_∞ algebras. We show how to pass from A_∞ to L_∞ algebras and therefore how to give an L_∞ presentation of open string field theory. At the quantum level, the L_∞ structure of the classical theory is modified to give a new structure that has been called a ”quantum L_∞ algebra.” Open-closed structures show a nontrivial interplay between the A_∞ and L_∞ structures, with open-closed vertices allowing one to interpret the open string fields as spaces carrying a representation of the L_∞ algebra, similar to the way in which matter fields can represent the action of diffeomorphisms in general relativity. Finally, we give the homotopy algebra background to the idea of Wilsonian effective actions. The tool, called ”homotopy transfer,” which we explain both for A_∞ and L_∞ structures, has had a number of applications. The second, section ”String Vertices”, focuses on the moduli spaces $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n punctures and the bundle $\widehat{\mathcal{P}}_{g,n}$ that includes local coordinates, defined up to a constant phase about each of the punctures. The constraint on the string vertices is the geometric version of the BV master equation and consistent vertices generate the correct string amplitudes. We review both the string vertices that arise from a minimal area problem, resulting at genus zero in surfaces build with flat metrics and curvature singularities, and the string vertices that can be defined via surfaces with hyperbolic metrics. We discuss an approach to open superstring theory in which the vertices incorporate canonical insertions of picture changing operators (PCO's).

> 虽然“玻色弦场论与超弦场论”和“弦场论的性质”几节回顾了弦场论的实用研究方法，但“弦场论的基础代数结构”和“弦顶点”两节深入探讨了构造弦场论所需的基础数学工具。其中第一节“弦场论的基础代数结构”梳理了开弦理论的 A_∞ 构造。对于闭弦，我们在经典层面建立了 L_∞ 代数的结构。我们说明了如何从 A_∞ 代数过渡到 L_∞ 代数，进而给出开弦场论的 L_∞ 表述。在量子层面，经典理论的 L_∞ 结构会发生修改，得到名为“量子 L_∞ 代数”的新结构。开-闭结构体现了 A_∞ 结构与 L_∞ 结构之间不平凡的相互作用: 借助开-闭顶点，可将开弦场解读为承载 L_∞ 代数表示的空间，这与广义相对论中物质场可以表示微分同胚作用的方式类似。最后，我们给出了威尔逊有效作用量思想的同伦代数背景。我们称之为“同伦转移”的工具可应用于 A_∞ 结构与 L_∞ 结构，目前已有诸多应用。第二节“弦顶点”重点讨论亏格为 g 、带有 n 个 puncture 的黎曼曲面的模空间 $\mathcal{M}_{g,n}$ ，以及包含每个 puncture 邻域内相差常数相位定义的局部坐标的丛 $\widehat{\mathcal{P}}_{g,n}$ 。弦顶点满足的约束是 BV 主方程的几何版本，自洽的顶点可以生成正确的弦振幅。我们回顾了两类弦顶点: 一类源自最小面积问题，在零亏格下得到由平坦度量和曲率奇点构造的曲面；另一类可通过双曲度量曲面定义。我们还讨论了开超弦场论的一种研究方案，该方案的顶点中包含了图象变换算子 (PCO) 的正则插入。

Superstring theory in the so-called ”large” Hilbert space of the superghost CFT is the subject of section ”Superstring Field Theories in the Large Hilbert Space”. The large Hilbert space is one that includes the zero mode ξ_0 of the ξ field in the construction of states. The ”small” Hilbert space, which does not contain such anticommuting zero mode, is the conventional state space of the theory, and all states are annihilated by η_0 , the anticommuting zero mode of the η field, with $\{\eta_0, \xi_0\} = 1$. We focus on the NS sector of open superstring, the formulation that was developed in [23,24]. The simplicity of the action, which takes a Wess-Zumino-Witten (WZW) form, lies in that no insertion of picture changing operators is needed. The string field is of picture number zero, and the required picture number is supplied by η_0 (of picture number minus one) that enters the action in the same way as the BRST operator does. The NS sector of heterotic strings also admits a relatively simple construction in the large Hilbert space, using as a building block a pure gauge field of bosonic closed string field theory [25,26]. For type II superstrings, a construction of the NS-NS sector has been given in [27]. This work uses as an ingredient a pure gauge field of NS heterotic strings in an L_∞ formulation in the small Hilbert space. Such a formulation, with canonical insertion of PCOs, has been given in [28] - the simpler case of an A_∞ canonical construction of open string field theory is reviewed here. We briefly consider the inclusion of the Ramond sector in this formalism [29, 30].

> “大希尔伯特空间中的超弦场论”一节讨论超鬼共形场论的所谓“大”希尔伯特空间中的超弦理论。大希尔伯特空间是在构造态时包含 ξ 场零模 ξ_0 的希尔伯特空间。不包含该反对易零模的“小”希尔伯特空间是该理论的传统态空间，其中所有态都会被 η_0 (η 场的反对易零模，满足 $\{\eta_0, \xi_0\} = 1$) 湮灭。我们重点研究开超弦的 NS 区，这是文献 [23,24] 中发展出的表述。该作用量具有 Wess-Zumino-Witten(WZW) 形式，其简洁性体现在不需要插入图象变换算子。弦场的图象数为零，所需的图象数由 η_0 (图象数为-1) 提供，η_0 以和 BRST 算子相同的方式进入作用量。杂化弦的 NS 区也可以在大希尔伯特空间中得到相对简洁的构造，以玻色闭弦场论的纯规范场作为构造模块 [25,26]。对于 II 型超弦，文献 [27] 已经给出了 NS-NS 区的构造。该工作将小希尔伯特空间 L_∞ 表述中杂化弦 NS 区的纯规范场作为基础组件。这种带有 PCO 正则插入的表述已在文献 [28] 中给出——本文也回顾了更简单的开弦场论 A_∞ 正则构造的情况。我们还简要讨论了如何在该形式体系中纳入拉蒙德区 [29, 30]。

Section ”Applications of String Field Theory” discusses some applications of string field theory. We examine tachyon condensation solutions-both numerical solutions using level truncation and analytical solutions using the picture of dressed surface states and the so-called Kbc algebra. We show how string field theory deals with mass renormalization and vacuum shifts. We give an introduction to the subject of D-instantons, as they are dealt within string field theory. This is also a subject where first-quantization cannot resolve the ambiguities in D-instanton corrections to perturbative scattering amplitudes. Finally, we review the analysis of the unitarity of the string theory S-matrix and explain the way in which string field theory makes a precise claim on the finiteness of string theory. This review concludes with section ”Some Future Directions”, where we discuss what may be fruitful research directions in string field theory.

节“弦场论的应用”将讨论弦场论的若干应用。我们考察快子凝聚解——包括利用能级截断的数值解，以及利用修饰表面态图像和所谓 Kbc 代数的解析解。我们展示弦场论如何处理质量重正化和真空位移。我们介绍弦场论框架下研究的 D 瞬子课题；该课题中，一阶量子化无法解决微扰散射振幅的 D 瞬子修正中存在的歧义。最后，我们回顾对弦论 S 矩阵幺正性的分析，阐释弦场论为何能对弦论的有限性给出明确论断。本综述以“若干未来研究方向”一节收尾，我们在其中探讨弦场论中可能富有成果的研究方向。

Throughout (most of) this review, we shall work setting $\alpha' = 1$.

在本综述的 (大部分) 内容中，我们将工作在 $\alpha' = 1$ 的设定下。

Glossary of Symbols

符号术语表

In this section, we give a glossary of widely used symbols.

本节给出常用符号术语表。

- $|0\rangle$SL $(2,\mathbb{C})$ invariant vacuum for closed strings, SL $(2,\mathbb{R})$ invariant vacuum for open strings.

  - $|0\rangle$SL $(2,\mathbb{C})$ 闭弦的不变真空，SL $(2,\mathbb{R})$ 开弦的不变真空。

- $\langle\cdots\rangle'$ correlator normalized as in (11)

  - $\langle\cdots\rangle'$ 按式 (11) 归一化的关联函数

- $\{\cdot,\cdot\}$ Batalin-Vilkovisky antibracket (96), antibracket for general complex (404)

  - $\{\cdot,\cdot\}$ 巴塔林-维尔可夫斯基反括号 (96)，一般复形的反括号 (404)

- $\{\cdot,\cdot\}_c$ antibracket on moduli spaces via closed strings (104),

  - $\{\cdot,\cdot\}_c$ 模空间上通过闭弦定义的反括号 (104),

- $\{\cdot,\cdot\}_o$ antibracket on moduli spaces via open strings (137),(156)

- $\{\cdot,\cdot\}_o$ 模空间上通过开弦定义的反括号 (137)、(156)

- $\{\cdot,\cdot,\cdots,\cdot\}$ multilinear function of string fields (107),(138),(157)

- $\{\cdot,\cdot,\cdots,\cdot\}$ 弦场的多重线性函数 (107)、(138)、(157)

- $[\cdot,\cdot,\cdots;\cdot,\cdot,\cdots,\cdot]_c$ multilinear product of string fields giving a closed string state (161)

- $[\cdot,\cdot,\cdots;\cdot,\cdot,\cdots,\cdot]_c$ 弦场的多重线性乘积，给出闭弦态 (161)

- $[\cdot,\cdot,\cdots;\cdot,\cdot,\cdots,\cdot]_o$ multilinear product of string fields giving an open string state (161)

- $[\cdot,\cdot,\cdots;\cdot,\cdot,\cdots,\cdot]_o$ 弦场的多重线性乘积，给出开弦态 (161)

- $[\cdot,\cdot,\cdots,\cdot]_c$ either $[\cdot,\cdot,\cdots;\cdot,\cdot,\cdots,\cdot]_c$ or $[\cdot,\cdot,\cdots;\cdot,\cdot,\cdots,\cdot]_o$ for only closed or only open strings

- $[\cdot,\cdot,\cdots,\cdot]_c$ 仅存在闭弦时为 $[\cdot,\cdot,\cdots;\cdot,\cdot,\cdots,\cdot]_c$ ，仅存在开弦时为 $[\cdot,\cdot,\cdots;\cdot,\cdot,\cdots,\cdot]_o$

- $b_0^\pm \equiv b_0 \pm \bar{b}_0$

- $|\mathbf{b}\rangle$ state inserted to create a boundary (81) |

- $\mathcal{B}\left[\frac{\partial}{\partial u^i}\right]$ antighost insertion (43),(66)

- $\mathcal{B}\left[\frac{\partial}{\partial u^i}\right]$ 反鬼插入 (43)、(66)

- b nilpotent operator in A_∞ and L_∞ algebras (314),(315),(338),(344)

- b A_∞ 和 L_∞ 代数中的幂零算子 (314)、(315)、(338)、(344)

- B the integral of the b -ghost field over the left half of the open string (559)

- B b 鬼场在开弦左半部分的积分 (559)

- $c_0^\pm \equiv \frac{1}{2}(c_0 \pm \bar{c}_0)$

- $d_{g,n}$ real dimension of $\mathcal{M}_{g,n}$ (39)

- $d_{g,n}$ $\mathcal{M}_{g,n}$ (39) 的实维数

- d_{g,b,n_c,n_o} real dimension of $\mathcal{M}_{g,b,n_c,n_o}$ (61)

- d_{g,b,n_c,n_o} $\mathcal{M}_{g,b,n_c,n_o}$ (61) 的实维数

- $\chi_{g,n}$ Euler number of surfaces of type (g,n) (40)

- $\chi_{g,n}$ (g,n) (40) 型曲面的欧拉示性数

- χ_{g,b,n_c,n_o} Euler number of surfaces of type (g,b,n_c,n_o) (62)

- χ_{g,b,n_c,n_o} (g,b,n_c,n_o) (62) 型曲面的欧拉示性数

- Δ Batalin-Vilkovisky delta operator (96), (402), (411) more generally

- Δ 巴塔林-维尔可夫斯基 δ 算子，更广泛定义见 (96)、(402)、(411)

- Δ_c delta operator for moduli spaces of surfaces via closed strings (104)

- Δ_c 闭弦模空间的 δ 算子 (104)

- Δ_o delta operator for moduli spaces of surfaces via open strings on different boundaries (156)

- Δ_o 开弦在不同边界上曲面模空间的 δ 算子 (156)

- Δ_o' delta operator for moduli spaces via open strings on the same boundary (156)

- Δ_o' 开弦在同一边界上模空间的 δ 算子 (156)

- $\eta^{\mu\nu} = \mathrm{diag}(-1,1,\cdots,1)$ Minkowski metric

- $\eta^{\mu\nu} = \mathrm{diag}(-1,1,\cdots,1)$ 闵可夫斯基度规

- $\eta_c = -\frac{1}{2\pi i}$ constant in the normalization of forms.

- $\eta_c = -\frac{1}{2\pi i}$ 形式归一化中的常数

- $\mathcal{F}_{g,n}$ subspace of $\widehat{\mathcal{P}}_{g,n}$ usually with $\mathcal{F}_{g,n} \to \mathcal{M}_{g,n}$ a map of degree one.

- $\mathcal{F}_{g,n}$ $\widehat{\mathcal{P}}_{g,n}$ 的子空间，通常 $\mathcal{F}_{g,n} \to \mathcal{M}_{g,n}$ 是一次映射

- $\mathcal{F}^s_{g,n}$ subspace of $\widetilde{\mathcal{P}}^s_{g,n}$ usually with $\mathcal{F}^s_{g,n} \to \mathcal{M}_{g,n}$ a map of degree one.

- $\mathcal{F}^s_{g,n}$ $\widetilde{\mathcal{P}}^s_{g,n}$ 的子空间，通常 $\mathcal{F}^s_{g,n} \to \mathcal{M}_{g,n}$ 是一次映射

- $\mathcal{F}^s_{g,b,n_c,n_o}$ subspace of $\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$ usually with $\mathcal{F}^s_{g,b,n_c,n_o} \to \mathcal{M}_{g,b,n_c,n_o}$ a map of degree one.

- $\mathcal{F}^s_{g,b,n_c,n_o}$ $\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$ 的子空间，通常 $\mathcal{F}^s_{g,b,n_c,n_o} \to \mathcal{M}_{g,b,n_c,n_o}$ 是一次映射

- g_o open string coupling (138)

- g_o 开弦耦合常数 (138)

- g_s string coupling (107)

  - g_s 弦耦合常数 (107)

- $\mathcal{H}_c$ state space of closed string theory (100),(122),(132),(155)

  - $\mathcal{H}_c$ 闭弦理论的态空间 (100)、(122)、(132)、(155)

- $\widetilde{\mathcal{H}}_c$ auxiliary state space of closed string theory (123),(133)

  - $\widetilde{\mathcal{H}}_c$ 闭弦理论的辅助态空间 (123)、(133)

- $\mathcal{H}_o$ state space of open string theory (152)

  - $\mathcal{H}_o$ 开弦理论的态空间 (152)

- $\widetilde{\mathcal{H}}_o$ auxiliary state space of open string theory (153)

  - $\widetilde{\mathcal{H}}_o$ 开弦理论的辅助状态空间 (153)

- h_b, h_f the L_0 eigenvalues of bosonic, fermionic open string states on D-instanton (590)

  - h_b, h_f D 瞬子上玻色、费米开弦态的 L_0 本征值 (590)

- $\mathcal{I}$ identity string field Ω_0

  - $\mathcal{I}$ 恒等弦场 Ω_0

- K normalization constant (12) of the open string vacuum, related to D-brane tension (255)

  - K 开弦真空的归一化常数 (12)，与 D 膜张力相关 (255)

- $\mathcal{K}$ the integral of the stress tensor over the left half of the open string (557)

  - $\mathcal{K}$ 应力张量在开弦左半部分的积分 (557)

- $L_0^\pm \equiv L_0 \pm \bar{L}_0$

- L same as **b** for L_∞ algebra

  - L 与 **b** 含义相同，用于 L_∞ 代数

- ℓ same as $\mathbf{L} - Q$

  - ℓ 与 $\mathbf{L} - Q$ 含义相同

- $\mathcal{M}_{g,n}$ moduli space of Riemann surfaces of genus g with n punctures

- $\mathcal{M}_{g,n}$ 带 n 个孔、亏格为 g 的黎曼曲面模空间

- $\overline{\mathcal{M}}_{g,n}$ Deligne-Mumford compactification of $\mathcal{M}_{g,n}$

- $\overline{\mathcal{M}}_{g,n}$ $\mathcal{M}_{g,n}$ 的德利涅-芒福德紧化

- $\mathcal{M}_{g,b,n_c,n_o}$ moduli space of surfaces of genus g, b boundaries, n_c/n_o closed/open string punctures

- $\mathcal{M}_{g,b,n_c,n_o}$ 带 g, b 个边界、n_c/n_o 个闭/开弦孔的曲面模空间

- M same as b for A_∞ algebra

- M 与 b 含义相同，用于 A_∞ 代数

- $\mathbf{m}$ same as $\mathbf{M} - Q$

- $\mathbf{m}$ 与 $\mathbf{M} - Q$ 含义相同

- N_{g,b,n_c,n_o} normalization constant for open-closed string forms (69),(70),(79)

- N_{g,b,n_c,n_o} 开-闭弦形式的归一化常数 (69),(70),(79)

- N exponential of the annulus partition function on the D-instanton (590),(592), (596)

- N D 瞬子上环面配分函数的指数 (590),(592), (596)

- $\mathcal{N}$ overall normalization for the D-instanton amplitudes given by $e^{-\mathcal{T}}N$

- $\mathcal{N}$ D 瞬子振幅的整体归一化，由 $e^{-\mathcal{T}}N$ 给出

- $\Omega_p^{(g,n)}$ p -forms on $\widehat{\mathcal{P}}_{g,n}$ or $\widehat{\mathcal{P}}^s_{g,n}$ (44) , (58)

- $\Omega_p^{(g,n)}$ p $\widehat{\mathcal{P}}_{g,n}$ 或 $\widehat{\mathcal{P}}^s_{g,n}$ (44) , (58) 上的-形式

- $\Omega_p^{(g,b,n_c,n_o)}$ p -forms on $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ or $\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$ (69) (79)

- $\Omega_p^{(g,b,n_c,n_o)}$ p - $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 或 $\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$ (69) (79) 上的形式

- $\widehat{\Omega}_p^{(g,b,n_c,n_o)}$ canonically normalized forms on $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ (68)

- $\widehat{\Omega}_p^{(g,b,n_c,n_o)}$ 上的典范归一化形式，定义于 $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ (68)

- $|\Omega_\alpha\rangle$ wedge state of width α Fig. 17

- $|\Omega_\alpha\rangle$ 宽度为 α 的楔形态图 17

- $\widehat{\mathcal{P}}_{g,n}$ bundle over $\mathcal{M}_{g,n}$ with fibers giving local coordinates up to phases.
    - $\widehat{\mathcal{P}}_{g,n}$ 是 $\mathcal{M}_{g,n}$ 上的丛，其纤维在相差一个相位的意义下给出局部坐标。
- $\widehat{\mathcal{P}}^{s}_{g,n}$ bundle over $\widehat{\mathcal{P}}_{g,n}$ with fibers giving locations of PCO’s.
    - $\widehat{\mathcal{P}}^{s}_{g,n}$ 丛，以 PCO 位置为纤维，底空间为 $\widehat{\mathcal{P}}_{g,n}$ 。
- $\widehat{\mathcal{P}}_{g,b}$, bundle over $\mathcal{M}_{g,b,n_c,n_o}$ with fibers giving local coordinates up to phases.
    - $\widehat{\mathcal{P}}_{g,b}$ ， $\mathcal{M}_{g,b,n_c,n_o}$ 上的丛，其纤维给出相差相位的局部坐标。
- $\widehat{\mathcal{P}}^{s}_{g,b,n_c,n_o}$ bundle over $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ with fibers giving positions for PCO’s.
    - $\widehat{\mathcal{P}}^{s}_{g,b,n_c,n_o}$ 丛，其纤维为 PCO 提供位置，底空间为 $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 。
- Q BRST operator (1), (23), (182), (203)
    - Q BRST 算符 (1), (23), (182), (203)
- $\Psi/\widetilde{\Psi}$ closed string field/auxiliary field (97),(122),(123),(132),(133)
    - $\Psi/\widetilde{\Psi}$ 闭弦场/辅助场 (97),(122),(123),(132),(133)
- $\Psi_o/\widetilde{\Psi}_o$ open string field in open-closed theory/auxiliary field (155)
    - $\Psi_o/\widetilde{\Psi}_o$ 开-闭弦理论中的开弦场/辅助场 (155)
- $\psi_o/\widetilde{\psi}_o$ open string field for classical bosonic theory (138) and classical open superstrings (154)
    - $\psi_o/\widetilde{\psi}_o$ 经典玻色理论 (138) 与经典开超弦 (154) 的开弦场
- $\mathcal{T}$ D-brane tension (140), relation to K (255)
    - $\mathcal{T}$ D 膜张力 (140)，与 K 的关系 (255)
- $T(V)$ tensor co-algebra for A_∞ algebras (312)
    - $T(V)$ 弦场论中 A_∞ 代数的张量余代数 (312)
- $T(W)$ symmetrized tensor co-algebra for L_∞ algebras (341)
    - $T(W)$ L_∞ 代数的对称张量余代数 (341)
- $\mathcal{V}_{g,n}$ string vertices for closed string theory (105)

- $\mathcal{V}_{g,n}$ 闭弦理论的弦顶点 (105)

- $\mathcal{V}_{g,b,n_c,n_o}$ string vertices for open-closed string theory (456)
  - $\mathcal{V}_{g,b,n_c,n_o}$ 开-闭弦理论的弦顶点 (456)
- $\mathcal{X}(z), \overline{\mathcal{X}}(\bar{z})$ Grassmann even PCO's (24)
  - $\mathcal{X}(z), \overline{\mathcal{X}}(\bar{z})$ 格拉斯曼偶 PCO(24)
- $\mathcal{X}_0, \overline{\mathcal{X}}_0$ zero modes of PCO's (125)
  - $\mathcal{X}_0, \overline{\mathcal{X}}_0$ PCO 的零模 (125)

Background Material

背景材料

In this section, we shall review some background information on world-sheet string theory, string amplitudes, and their off-shell generalization. We do this both for bosonic strings and for superstrings. This includes the definition of suitable forms for integration over the moduli spaces of Riemann surfaces, the subtleties that arise due to the inclusion of picture changing operators in superstring theory, and the issues with normalization and signs of open-closed string amplitudes. We also review the Batalin-Vilkovisky (BV) formalism that will be useful in the formulation of string field theory.

在本节中，我们将回顾世界面弦论、弦振幅及其脱壳推广的相关背景知识，covering 玻色弦与超弦两类情形。内容包括黎曼曲面模空间上合适积分形式的定义、超弦理论中因引入图像变换算子产生的微妙问题，以及开弦-闭弦振幅的归一化与符号问题。我们还会回顾巴塔林-维尔可夫斯基 (BV) 形式体系，该体系对弦场论的构建十分有用。

World-Sheet Conventions

世界面约定

In this subsection, we shall describe the world-sheet conventions that we shall use in our analysis. The world-sheet of bosonic string theory has a matter conformal field theory (CFT) of central charge 26 and $b, c, \bar{b}, \bar{c}$ ghost system carrying total central charge -26. The BRST operator is given by

在本小节中，我们将描述分析过程中使用的世界面约定。玻色弦理论的世界面包含中心荷为 26 的物质共形场论 (CFT)，以及总中心荷为-26 的 $b, c, \bar{b}, \bar{c}$ 鬼系统。BRST 算符由下式给出

$$Q = \oint_0 j_B(z)\, dz + \oint_0 \bar{j}_B(\bar{z})\, d\bar{z} \tag{1}$$

where

其中

$$j_B = cT_m + bc\partial c,\ \bar{j}_B = \bar{c}\bar{T}_m + \bar{b}\bar{c}\bar{\partial}\bar{c}, \tag{2}$$

with the products $bc\partial c$ and $\bar{b}\bar{c}\bar{\partial}\bar{c}$ appropriately normal ordered. Here, $T_m, \bar{T}_m$ denote the matter stress tensor, and $\oint_0$ denotes integration around the closed contour around the origin, normalized so that

乘积 $bc\partial c$ 和 $\bar{b}\bar{c}\bar{\partial}\bar{c}$ 已适当取正规序。此处，$T_m, \bar{T}_m$ 表示物质能量动量张量，$\oint_0$ 表示绕原点闭合围道的积分，归一化满足

$$\oint_0 \frac{dz}{z} = 1, \oint_0 \frac{d\bar{z}}{\bar{z}} = 1. \tag{3}$$

Using the operator product expansions of the ghost fields,

利用鬼场的算符乘积展开，

$$b(z)\,c(w) \simeq \frac{1}{z-w},\ \bar{b}(\bar{z})\,\bar{c}(\bar{w}) \simeq \frac{1}{\bar{z}-\bar{w}}, \tag{4}$$

and those of the matter stress tensor,

以及物质能量动量张量的算符乘积展开，

$$\begin{aligned} T_m(z)\,T_m(w) &\simeq \frac{26}{2}\frac{1}{(z-w)^4} + \frac{2}{(z-w)^2}T_m(w) + \frac{1}{z-w}\partial_w T_m(w), \\ \bar{T}_m(\bar{z})\,\bar{T}_m(\bar{w}) &\simeq \frac{26}{2}\frac{1}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2}\bar{T}_m(\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\bar{\partial}_w \bar{T}_m(\bar{w}), \end{aligned} \tag{5}$$

one can show that Q defined in (1) squares to zero: $Q^2 = \frac{1}{2}\{Q,Q\} = 0$. The ghost conformal field theory also has stress tensors T_{bc} and $\bar{T}_{\bar{b}\bar{c}}$ given by

可以证明 (1) 式中定义的 Q 平方为零: $Q^2 = \frac{1}{2}\{Q,Q\} = 0$ 。鬼共形场论也存在能量动量张量 T_{bc} 和 $\bar{T}_{\bar{b}\bar{c}}$ ，由下式给出

$$T_{bc}(z) = -\partial bc - 2b\partial c,\ \bar{T}_{bc}(\bar{z}) = -\bar{\partial}\bar{b}\bar{c} - 2\bar{b}\bar{\partial}\bar{c}. \tag{6}$$

Although for much of our analysis we shall not need the form of the matter part of the world-sheet CFT, for a space-time interpretation, we shall take the theory to be a direct sum of D free scalars X^μ describing flat, non-compact directions and a CFT of central charge $(26-D)$. The X^μ ’ s satisfy the operator product expansion,

虽然大部分分析并不需要世界面 CFT 物质部分的具体形式，但为了给出时空诠释，我们将该理论取为 D 个描述平坦非紧致方向的自由标量 X^μ 与一个中心荷为 $(26-D)$ 的 CFT 的直和。这些 X^μ 满足如下算符乘积展开，

$$\partial X^\mu(z)\,\partial X^\upsilon(w)=-\frac{\eta^{\mu\upsilon}}{2(z-w)^2},\ \bar{\partial}X^\mu(\bar{z})\,\bar{\partial}X^\upsilon(\bar{w})=-\frac{\eta^{\mu\upsilon}}{2(\bar{z}-\bar{w})^2}. \tag{7}$$

We work in the mostly plus convention with $\eta^{\mu\upsilon}=\mathrm{diag}\,(-1,1,\cdots,1)$. They contribute to the matter stress tensor as follows:

我们采用符号差大多数为正的约定，即 $\eta^{\mu\upsilon}=\mathrm{diag}\,(-1,1,\cdots,1)$ 。它们对物质能量动量张量的贡献如下:

$$T(z)=-\eta_{\mu\nu}\partial X^\mu\partial X^\nu,\ \bar{T}(\bar{z})=-\eta_{\mu\nu}\bar{\partial}X^\mu\bar{\partial}X^\nu. \tag{8}$$

We can now define the "vacuum" carrying momentum k via

我们现在可以通过下式定义携带动量 k 的“真空”

$$|k\rangle\equiv e^{ik.X}(0)|0\rangle, \tag{9}$$

where $|0\rangle$ is the $\mathrm{SL}(2,\mathbb{C})$ invariant vacuum. We also have the out vacuum with momentum k given by $\langle k|\equiv\langle 0\mid I_c\circ e^{ik.X}(0)$, where $I_c(z)=1/z$ is the BPZ conjugation operator for closed strings, and for any function $f(z)$ and operator O , we write $f\circ O$ for the conformal transform of O under f . We shall normalize the $\mathrm{SL}(2,\mathbb{C})$ invariant vacuum such that

其中 $|0\rangle$ 是 $\mathrm{SL}(2,\mathbb{C})$ 不变真空。我们还有携带动量 k 的外真空，由 $\langle k|\equiv\langle 0\mid I_c\circ e^{ik.X}(0)$ 给出，其中 $I_c(z)=1/z$ 是闭弦的 BPZ 共轭算符，并且对任意函数 $f(z)$ 和算符 O , 我们用 $f\circ O$ 表示 O 在 f 下的共形变换。我们对 $\mathrm{SL}(2,\mathbb{C})$ 不变真空的归一化满足

$$\langle k|c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1|k'\rangle=-(2\pi)^D\delta^{(D)}(k+k'). \tag{10}$$

Due to the state/operator correspondence, there is one-to-one correspondence between the states $|\phi\rangle$ of the CFT and the local operators $\phi(z,\bar{z})$ in the CFT via $|\phi\rangle=\phi(0)|0\rangle$. Oftentimes, the symbol Ψ may be used interchangeably to denote a state or a local operator in the CFT.

由于态/算符对应，CFT 中的态 $|\phi\rangle$ 与 CFT 中的局域算符 $\phi(z,\bar{z})$ 通过 $|\phi\rangle=\phi(0)|0\rangle$ 一一对应。通常符号 Ψ 可互换用来表示 CFT 中的一个态或一个局域算符。

If the background in which we formulate string theory contains D-branes, then the spectrum also contains open string states. In this case, the world-sheet may also contain boundaries lying on the D-branes. The open string states are in one-to-one correspondence to local operators on the boundary of the world-sheet. The BRST operator for open string is obtained by restricting the integrals in (1) to run over the semi-circles lying in the upper half plane. Using the doubling trick that maps the anti-holomorphic fields in the upper half plane to holomorphic fields in the lower half plane, it can be expressed as only the first term on the right-hand side of (1). In the upper-half plane the boundary condition relates the oscillators of the anti-holomorphic field to those of the holomorphic fields.

如果我们构造弦理论的背景中包含 D 膜，那么谱中还会包含开弦态。这种情况下，世界面也可以存在位于 D 膜上的边界。开弦态与世界面边界上的局域算符一一对应。开弦的 BRST 算符可以通过将 (1) 式中的积分限制在上半平面的半圆周上得到。利用将上半平面的反全纯场映射到下半平面的全纯场的加倍技巧，该算符可以仅表示为 (1) 式右侧的第一项。在上半平面中，边界条件将反全纯场的振子与全纯场的振子关联起来。

When dealing with open string amplitudes on a disk, we can normalize the $\mathrm{SL}(2,\mathbb{R})$ invariant open string vacuum so that

处理圆盘上的开弦振幅时，我们可以对 $\mathrm{SL}(2,\mathbb{R})$ 不变开弦真空归一化，使得

$$\langle k|c_{-1}c_0c_1|k'\rangle' = -(2\pi)^{p+1}\delta^{(p+1)}(k+k'), \tag{11}$$

where $p+1$ is the number of non-compact space-time directions spanned by the D-brane and now $\langle k| = \langle 0 \mid I_o \circ e^{ik.X}(0)$ where $I_o(z) = -1/z$. We added the superscript $'$ for the following reason. The normalization chosen here is convenient to write classical open string field theory, but when considering the string field theory of open and closed strings, it will be more natural to use a different normalization. To see this, consider the closed string one-point function on the disk using the normalization given in (11). The result for the one-point function of D-dimensional gravitons will be insensitive to the boundary condition along the compact directions since the graviton vertex operator only involves the scalar fields of the non-compact coordinates. On the other hand, this one point function is expected to be governed by the D-brane tension, which depends on the boundary condition along the compact direction, e.g., our Dp brane with Neumann or Dirichlet boundary conditions along a compact dimension is a different D-brane with different tensions and must give different results for the graviton one-point function. For this reason, to construct open-closed string field theory, we introduce a new inner product where we multiply the right-hand side of (11) by a constant K that is to be determined in terms of the D-brane tension:

其中 $p+1$ 是 D 膜张成的非紧致时空方向的数目，此时 $\langle k| = \langle 0 \mid I_o \circ e^{ik.X}(0)$，且 $I_o(z) = -1/z$。我们加上标 $'$ 的原因如下。这里选取的归一化便于写出经典开弦场论，但在考虑开弦与闭弦的弦场论时，使用不同的归一化会更自然。对此，我们利用 (11) 式给出的归一化考察圆盘上的闭弦单点函数。D 维引力子单点函数的结果不依赖紧致方向的边界条件，因为引力子顶点算符仅涉及非紧致坐标的标量场。另一方面，该单点函数预期由 D 膜张力决定，而张力依赖于紧致方向的边界条件，例如，我们的 Dp 膜在紧致维度上取诺依曼或狄利克雷边界条件时，对应不同的 D 膜，具有不同的张力，得到的引力子单点函数结果也不同。因此，为构造开-闭弦场论，我们引入新的内积，将 (11) 式的右侧乘以一个待由 D 膜张力确定的常数 K：

$$\langle k|c_{-1}c_0c_1|k'\rangle = -(2\pi)^{p+1}K\delta^{(p+1)}(k+k'). \tag{12}$$

This overlap has no prime superscript. The precise relation between K and the D-brane tension will be discussed in section ”Results for Special Amplitudes”. While discussing disk correlation functions, we shall use the $\langle\rangle'$ if we use the normalization (11) and $\langle\rangle$ if we use the normalization (12).

这个重叠没有原上标。K 与 D 膜张力的精确关系将在“特殊振幅的结果”一节讨论。讨论圆盘关联函数时，若我们使用归一化 (11) 就采用 $\langle\rangle'$ ，若使用归一化 (12) 就采用 $\langle\rangle$ 。

Since $Q^2 = 0$, we can introduce the notion of BRST cohomology. We take the space of states that are annihilated by Q and declare two states to be equivalent if they differ by a state of the form $Q|s\rangle$, for some state $|s\rangle$. The states spanning such space are the elements of the BRST cohomology. For open bosonic strings, the physical states are in one-to-one correspondence with the elements of the BRST cohomology at ghost number one.

由于 $Q^2 = 0$ ，我们可以引入 BRST 上同调的概念。我们取被 Q 零化的态空间，若两个态相差一个形如 $Q|s\rangle$ 的态 (对某个态 $|s\rangle$)，则我们称这两个态等价。张成该空间的态就是 BRST 上同调的元素。对于开玻色弦，物理态与鬼数为 1 的 BRST 上同调元素一一对应。

For closed bosonic strings, the relevant state space is not the original CFT space $\mathcal{H}'$ but rather a restricted subspace $\mathcal{H}_c \subset \mathcal{H}'$ spanned by states satisfying the ”level-matching” conditions:

对于闭玻色弦，相关的态空间不是原始的 CFT 空间 $\mathcal{H}'$ ，而是满足“能级匹配”条件的态张成的受限子空间 $\mathcal{H}_c \subset \mathcal{H}'$:

$$|s\rangle \in \mathcal{H}_c \ \text{ iff } \ b_0^-|s\rangle = 0, \ L_0^-|s\rangle = 0, \tag{13}$$

where

其中

$$b_0^\pm = b_0 \pm \bar{b}_0, \ L_0^\pm = L_0 \pm \bar{L}_0, \ c_0^\pm = (c_0 \pm \bar{c}_0)/2. \tag{14}$$

These constraints are needed to define consistent off-shell amplitudes for the states. The physical closed string states are in one-to-one correspondence to the cohomology of Q in $\mathcal{H}_c$ at ghost number two. For generic momentum this agrees with the cohomology in the full unconstrained space $\mathcal{H}'$, but there are important differences at zero momentum. In particular, the zero-momentum dilaton is a cohomology class in $\mathcal{H}_c$ but happens to be trivial in $\mathcal{H}'$. For generic momentum, the representative elements of BRST cohomology can be taken to be of the form $c\bar{c}W$ where W is a dimension (1,1) primary operator in the matter CFT. It is not necessary to choose it this way, however, and the procedure for computing string theory amplitudes that we shall describe in section ”Bosonic String Amplitudes and Their Off-Shell Generalization” does not rely on this choice.

这些约束是定义态的一致 off-shell 振幅所必需的。物理闭弦态与 $\mathcal{H}_c$ 中 Q 在鬼数为 2 处的上同调一一对应。对于一般动量，这与全未约束空间 $\mathcal{H}'$ 中的上同调一致，但零动量处存在重要差异: 零动量 dilaton 是 $\mathcal{H}_c$ 中的上同调类，但在 $\mathcal{H}'$ 中是平凡上同调类。对于一般动量，BRST 上同调的代表元可以取为 $c\bar{c}W$ 的形式，其中 W 是物质 CFT 中维度为 (1,1) 的主算符。不过不必非要这样选取，我们在“玻色弦振幅及其 off-shell 推广”一节介绍的弦论振幅计算过程不依赖这一选取。

Recall that in any CFT we have the bilinear BPZ inner product. To define it, one needs the notion of the BPZ dual state: associated with a state $|B\rangle = B(0)|0\rangle$, obtained by acting with the $B(z)$ operator on the ”past”

SL$(2,\mathbb{C})$ vacuum $|0\rangle$ at $z=0$, we have the BPZ dual state $\langle B| \equiv \langle 0| I_c \circ B(0)$ acting on the "future" SL$(2,\mathbb{C})$ vacuum $\langle 0|$ at $z=\infty$, with insertion of the operator $B(z)$ using the inversion map $I_c(z)=1/z$. The bilinear BPZ inner product of two states, $|A\rangle$ and $|B\rangle$, is simply the overlap $\langle A \mid B\rangle$. In any conformal field theory, this inner product is non-degenerate: if $\langle A \mid B\rangle = 0$ for all $|A\rangle$, then $|B\rangle = 0$. In particular, the BPZ inner product is non-degenerate in $\mathcal{H}'$. For future reference, we note that for open strings, we have a similar definition of BPZ inner product except that the map $I_c(z)$ is replaced by $I_o(z) = -1/z$ in order to ensure that it is an $SL(2,R)$ transformation.

回顾可知，在任意共形场论 (CFT) 中都存在双线性 BPZ 内积。要定义它，我们需要 BPZ 对偶态的概念: 将 $B(z)$ 算子作用于"过去" SL$(2,\mathbb{C})$ 真空 $|0\rangle$ ，在 $z=0$ 处得到态 $|B\rangle = B(0)|0\rangle$ ，与之对应，我们将 BPZ 对偶态 $\langle B| \equiv \langle 0| I_c \circ B(0)$ 通过反演映射 $I_c(z)=1/z$ 插入算子 $B(z)$ ，作用于"未来" SL$(2,\mathbb{C})$ 真空 $\langle 0|$ ，在 $z=\infty$ 处得到该对偶态。两个态 $|A\rangle$ 和 $|B\rangle$ 的双线性 BPZ 内积就是交叠 $\langle A \mid B\rangle$ 。在任意共形场论中，这个内积都是非退化的: 即如果对所有 $|A\rangle$, then $|B\rangle = 0$ 都有 $\langle A \mid B\rangle = 0$ 。特别地，BPZ 内积在 $\mathcal{H}'$ 中是非退化的。为方便后续参考，我们注意，开弦的 BPZ 内积有类似定义，区别仅在于映射 $I_c(z)$ 被替换为 $I_o(z) = -1/z$ ， 以保证这是一个 $SL(2,R)$ 变换。

If both $|A\rangle$ and $|B\rangle$ are annihilated by b_0^- , the dual states $\langle A||$ and $\langle B|$ are also annihilated by b_0^- , and then we find $\langle A \mid B\rangle = 0$. Indeed, noticing that $\{c_0^-, b_0^-\} = 1$, we have $\langle A \mid B\rangle = \langle A|(c_0^- b_0^- + b_0^- c_0^-)|B\rangle = 0$, with the first term killing the ket and the second term killing the bra. This deficiency is fixed by defining an alternative bilinear inner product $\langle \cdot, \cdot \rangle$ suitable for $\mathcal{H}_c$:

如果两个 $|A\rangle$ and $|B\rangle$ 都被 b_0^- 零化, 那么对偶态 $\langle A||$ 和 $\langle B|$ 也会被 b_0^- 零化, 由此我们得到 $\langle A \mid B\rangle = 0$ 。事实上，注意到 $\{c_0^-, b_0^-\} = 1$ ， 我们有 $\langle A \mid B\rangle = \langle A|(c_0^- b_0^- + b_0^- c_0^-)|B\rangle = 0$ ， 其中第一项零化右矢，第二项零化左矢。我们可以通过定义一个适用于 $\mathcal{H}_c$ 的替代双线性内积 $\langle \cdot, \cdot \rangle$ 来解决这个缺陷:

$$\langle A, B\rangle \equiv \langle A| c_0^- |B\rangle \quad A, B \in \mathcal{H}_c. \tag{15}$$

Note that any $|A'\rangle \in \mathcal{H}'$ can be written as $|A'\rangle = |A\rangle + c_0^-|a\rangle$, with $|A\rangle, |a\rangle \in \mathcal{H}_c$. Moreover $\langle A', B\rangle = \langle A, B\rangle$, because the part of the state A' not annihilated by b_0^- drops out. We claim that the above inner product is non-degenerate in the constrained space $\mathcal{H}_c$. We see this as follows. Suppose $\langle A, B\rangle = 0$ for all $|A\rangle$ killed by b_0^- and for $b_0^-|B\rangle = 0$. By the above remark, this means that $\langle A', B\rangle = 0$ for all $|A'\rangle \in \mathcal{H}'$. By the non-degeneracy of the BPZ inner product on $\mathcal{H}'$, this implies $c_0^- |B\rangle = 0$. Acting on this with b_0^- and since $b_0^-|B\rangle = 0$, we conclude that $|B\rangle = 0$. This establishes the claimed non-degeneracy on $\mathcal{H}_c$.

注意任意 $|A'\rangle \in \mathcal{H}'$ 都可以写为 $|A'\rangle = |A\rangle + c_0^-|a\rangle$ ， 其中 $|A\rangle, |a\rangle \in \mathcal{H}_c$ 。此外 $\langle A', B\rangle = \langle A, B\rangle$ ， 这是因为态 A' 中不被 b_0^- 零化的部分会退耦合。我们断言上述内积在约束空间 $\mathcal{H}_c$ 上是非退化的。证明如下: 假设对于所有被 b_0^- 零化的 $|A\rangle$ 和 $b_0^-|B\rangle = 0$ ， 都有 $\langle A, B\rangle = 0$ 。根据上述结论，这意味着对所有 $|A'\rangle \in \mathcal{H}'$ 都有 $\langle A', B\rangle = 0$ 。由 BPZ 内积在 $\mathcal{H}'$ 上的非退化性可得 $c_0^- |B\rangle = 0$. Acting on this with b_0^- and since $b_0^-|B\rangle = 0$ ， 因此我们推出 $|B\rangle = 0$ 。这就证明了 $\mathcal{H}_c$ 上的非退化性。

For open strings, we do not need any c_0^- insertion in the definition of the inner product; it is simply the BPZ inner product. For uniformity of notation, however, we shall define

对于开弦，我们在内积的定义中不需要插入任何 c_0^-；它就是 BPZ 内积。但为了记号统一，我们仍会定义

$$\langle A, B\rangle' \equiv \langle A \mid B\rangle',\ \langle A, B\rangle \equiv \langle A \mid B\rangle. \tag{16}$$

It should be possible to tell from the context if one is doing open or closed string theory.

从上下文应当可以区分讨论的是开弦还是闭弦理论。

In the type IIA or IIB string theory, we also have $\beta, \gamma, \overline{\beta}, \overline{\gamma}$ bosonic ghosts, and the matter part of the world-sheet CFT has central charge 15. We shall describe some of the properties of the holomorphic sector of the world-sheet CFT, keeping in mind that an identical set of relations holds also for the anti-holomorphic fields. We can bosonize the β, γ system in terms of a pair of fermions (ξ, η) and a scalar ϕ as [31]

在 IIA 或 IIB 型弦论中，同样存在 $\beta, \gamma, \overline{\beta}, \overline{\gamma}$ 个玻鬼，世界面共形场论的物质部分中心荷为 15。我们将介绍世界面共形场论全纯 sector 的部分性质，需注意相同的关系对反全纯场同样成立。我们可以将 β, γ 系统玻色化，用一对费米子 (ξ, η) 和一个标量 ϕ 表示为 [31]

$$\gamma = \eta e^{\phi},\ \beta = \partial\xi e^{-\phi},\ \delta(\gamma) = e^{-\phi},\ \delta(\beta) = e^{\phi}. \tag{17}$$

We recall that the conformal dimensions of these fields are as follows:

我们回顾这些场的共形维数如下:

$$[\gamma] = -\frac{1}{2}, [\beta] = \frac{3}{2}, [\eta] = 1, [\xi] = 0, [\phi] = 0, \left[e^{q\phi}\right] = -\frac{1}{2}q(q+2). \tag{18}$$

The basic operator products are:

基本算符乘积为:

$$\xi(z)\eta(w) \simeq \frac{1}{z-w},\ \partial\phi(z)\partial\phi(w) \simeq -\frac{1}{(z-w)^2},$$

$$e^{q_1\phi}(z)e^{q_2\phi}(w) \simeq (z-w)^{-q_1q_2}e^{(q_1+q_2)\phi}(w). \tag{19}$$

The energy momentum tensor of the (β, γ) system is:

(β, γ) 系统的能量动量张量为:

$$T_{\beta\gamma} = \frac{3}{2}\beta\partial\gamma + \frac{1}{2}\gamma\partial\beta = T_{\xi\eta} + T_{\phi},\ T_{\xi\eta} = -\eta\partial\xi,\ T_{\phi} = -\frac{1}{2}\partial\phi\partial\phi - \partial^2\phi.$$

(20)

In the matter sector, besides the stress tensor T_m , we also have the dimension 3/2 superpartner T_F of the stress tensor, satisfying the operator product expansion:

在物质区，除了能量动量张量 T_m ，我们还有能量动量张量的维数为 3/2 的超伙伴 T_F ，满足算符乘积展开:

$$T_m(z)\,T_m(w) \simeq \frac{15}{2}\frac{1}{(z-w)^4} + \frac{2}{(z-w)^2}T_m(w) + \frac{1}{z-w}\partial_w T_m(w),$$

$$T_F(z)\,T_F(w) \simeq \frac{5}{2}\frac{1}{(z-w)^3} + \frac{1}{2}\frac{1}{z-w}T_m(w),$$

$$T_m(z)\,T_F(w) \simeq \frac{3}{2}\frac{1}{(z-w)^2}T_F(w) + \frac{1}{z-w}T_F(w). \tag{21}$$

When we have D non-compact flat directions, the matter sector contains D free bosons X^μ as in the case of bosonic string theory and also D free fermions ψ^μ with operator product expansion:

当存在 D 个非紧致平坦方向时，和玻色弦的情况一样，物质区包含 D 个自由玻色子 X^μ ，同时还有 D 个自由费米子 ψ^μ ，其算符乘积展开为:

$$\psi^\mu(z)\,\psi^\nu(w) \simeq -\frac{\eta^{\mu\nu}}{2(z-w)}. \tag{22}$$

The BRST current is modified to

BRST 流修正为

$$j_B = c\left(T_m + T_{\beta\gamma}\right) + bc\partial c + \gamma T_F - \frac{1}{4}\gamma^2 b. \tag{23}$$

The Grassmann even picture changing operator (PCO) field $\mathcal{X}(z)$ is defined as

格拉斯曼偶的图画改变算子 (PCO) 场 $\mathcal{X}(z)$ 定义为

$$\mathcal{X}(z) = \{Q, \xi(z)\} = c\partial\xi + e^{\phi}T_F - \frac{1}{4}\partial\eta e^{2\phi}b - \frac{1}{4}\partial\left(\eta e^{2\phi}b\right). \tag{24}$$

We assign various quantum numbers to the fields as follows:

我们给场分配各种量子数如下:

1. We assign ghost number to the fields as follows:

1. 我们给场分配鬼数如下:

$$\mathrm{gh}(c) = \mathrm{gh}(\eta) = \mathrm{gh}(\gamma) = 1, \tag{25}$$

$$\mathrm{gh}(b) = \mathrm{gh}(\xi) = \mathrm{gh}(\beta) = -1.$$

The matter fields X^μ, ψ^μ , as well as the ϕ field, have ghost number zero. Note that the PCO has ghost number zero, and so do the stress tensor T and the superpartner T_F . The assignment in the anti-holomorphic sector is analogous.

物质场 X^μ, ψ^μ 和 ϕ 场的鬼数均为零。注意图画改变算子的鬼数为零，能量动量张量 T 和其超伙伴 T_F 的鬼数也为零。反全纯区的分配是类似的。

2. We assign holomorphic picture number ”pic” as follows:

2. 我们分配全纯图画数“pic”如下:

$$\operatorname{pic}(\eta) = -1, \ \operatorname{pic}(\xi) = 1, \ \operatorname{pic}\left(e^{q\phi}\right) = q, \ \operatorname{pic}(\mathcal{X}) = 1. \tag{26}$$

All matter fields, as well as the (β, γ) ghosts, have zero picture number. The stress tensors T and the superpartners T_F also have zero picture number. There is separate anti-holomorphic picture number defined for the anti-holomorphic fields.

所有物质场与 (β, γ) 鬼场的图景数均为零。能量动量张量 T 与其超伙伴 T_F 的图景数也为零。我们还为反全纯场定义了独立的反全纯图景数。

3. Associated with the world-sheet fermion number F operator with integer eigenvalues, there is a GSO (Gliozzi-Scherk-Olive) operator $(-1)^F$. An operator $\mathcal{O}$ such that $(-1)^F \mathcal{O} = \mathcal{O}(-1)^F$ is said to be GSO even, and an operator for which $(-1)^F \mathcal{O} = -\mathcal{O}(-1)^F$ is said to be GSO odd. We have:

3. 与具有整数本征值的世界面费米子数 F 算符对应，存在 GSO(Gliozzi-Scherk-Olive) 算符 $(-1)^F$ 。满足 $(-1)^F \mathcal{O} = \mathcal{O}(-1)^F$ 的算符 $\mathcal{O}$ 称为 GSO 偶，满足 $(-1)^F \mathcal{O} = -\mathcal{O}(-1)^F$ 的算符称为 GSO 奇。我们有:

GSO odd fields: $\beta, \gamma, \psi^\mu, T_F$.

GSO 奇场: $\beta, \gamma, \psi^\mu, T_F$ 。

(27)

The operator $e^{q\phi}$ has GSO parity $(-1)^q$ for integer q , namely, $(-1)^F e^{q\phi} = (-1)^q e^{q\phi}(-1)^F$. The rest of the fields introduced above, including X^μ, c, b, η , $\xi, j_B, \mathcal{X}$, are GSO even.

对于整数 q ，算符 $e^{q\phi}$ 的 GSO 宇称为 $(-1)^q$ ，即 $(-1)^F e^{q\phi} = (-1)^q e^{q\phi}(-1)^F$ 。上文引入的其余场，包括 X^μ, c, b, η 、$\xi, j_B, \mathcal{X}$ ，均为 GSO 偶。

4. NS sector vertex operators are obtained by taking the product of $e^{q\phi}$ for $q \in \mathbb{Z}$ and a regular vertex operator constructed from the ghosts b, c, β, γ and the matter fields. Ramond sector vertex operators are obtained by taking the product of $e^{q\phi}$ for $q \in \mathbb{Z} + \frac{1}{2}$ and a ”spin field” whose operator product with the GSO even NS sector fields is single valued but whose operator product with GSO odd NS sector fields has square root branch point singularities. The GSO projection rules in the R sector have a twofold ambiguity since a given spin field may be declared to be GSO odd or even, but once a choice has been made for one such spin field, the GSO parity of the rest of the operators is chosen to be such that if V is a GSO even operator in the

R-sector, then the operator product of V with any GSO odd (even) operator in the NS sector will only generate GSO odd (even) operators in the R sector. Type IIA and IIB theories differ in the specification of which half of the states are declared as GSO even in the R sector.

4. NS 区顶点算符由 $q \in \mathbb{Z}$ 对应的 $e^{q\phi}$，与由鬼场 b, c, β, γ 和物质场构造的正则顶点算符相乘得到。拉蒙德区顶点算符由 $q \in \mathbb{Z} + \frac{1}{2}$ 对应的 $e^{q\phi}$ 与一个“旋量场”相乘得到，该旋量场与 GSO 偶 NS 区场的算符乘积是单值的，与 GSO 奇 NS 区场的算符乘积存在平方根分支点奇点。R 区的 GSO 投影规则存在两重歧义，因为任意给定旋量场既可以被定义为 GSO 奇也可以定义为 GSO 偶，但只要对一个这样的旋量场做出选择，其余算符的 GSO 宇称就会被确定: 若 V 是 R 区的一个 GSO 偶算符，那么 V 与 NS 区任意 GSO 奇 (偶) 算符的算符乘积只会生成 R 区的 GSO 奇 (偶) 算符。IIA 型与 IIB 型理论的区别在于 R 区中指定哪一半态为 GSO 偶。

In type IIA or type IIB superstring theories, we have separate GSO operators in the left- and right-moving sectors, and we require the state to be invariant under each of these GSO operators. There are also versions of superstring theory without spacetime supersymmetry, known as type 0A and type 0B theories, where we require the states to be invariant under the combined operation of left and right GSO operation. As in bosonic string theory, the physical states in all these theories are taken to be the BRST cohomology classes. The picture number, however, is fixed. We require picture -1 for the NS sector and picture -1/2 for R sector.

在 IIA 或 IIB 型超弦理论中，左行和右行区各有独立的 GSO 算符，我们要求态在每个 GSO 算符作用下保持不变。也存在不具有时空超对称的超弦理论版本，即 0A 型和 0B 型理论，这类理论要求态在左右 GSO 算符的组合操作下保持不变。与玻色弦理论一样，所有这些理论的物理态都取为 BRST 上同调类。但图景数是固定的: 我们要求 NS 区的图景数为-1，R 区的图景数为-1/2。

We define momentum states for closed and open strings, respectively,

我们分别对闭弦和开弦定义动量态,

$$\left|k; m, n\right\rangle \equiv e^{ik.X}(0)\, e^{m\phi}(0)\, e^{n\bar{\phi}}(0)\left|0\right\rangle,\ \left|k; m\right\rangle \equiv e^{ik.X}(0)\, e^{m\phi}(0)\left|0\right\rangle. \tag{28}$$

In the first definition, $|0\rangle$ is the $\mathrm{SL}(2,\mathbb{C})$ invariant closed string vacuum, while in the second definition, $|0\rangle$ is the $\mathrm{SL}(2,\mathbb{R})$ invariant open string vacuum. The analog of the normalization conditions (10) and (11) is:

第一种定义中，$|0\rangle$ 是 $\mathrm{SL}(2,\mathbb{C})$ 不变的闭弦真空；第二种定义中，$|0\rangle$ 是 $\mathrm{SL}(2,\mathbb{R})$ 不变的开弦真空。归一化条件对应于 (10) 和 (11) 的形式为:

$$\left\langle k; 0, 0\left|c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1e^{-2\phi}e^{-2\bar{\phi}}\right|k'; 0, 0\right\rangle = -(2\pi)^D\delta^{(D)}(k+k'), \tag{29}$$

and

且

$$\langle k; 0\left|c_{-1}c_0c_1e^{-2\phi}\right|k'; 0\rangle' = (2\pi)^{p+1}\delta^{(p+1)}(k+k'), \tag{30}$$

where the locations of the operators $e^{-2\phi}, e^{-2\overline{\phi}}$ on the world-sheet are irrelevant since they are dimension zero vertex operators. [2] Note the $'$ on the correlator in (30); in open closed string field theory, we have to define a more general inner product analogously to (12) with a constant K on the right-hand side.

其中算符 $e^{-2\phi}, e^{-2\overline{\phi}}$ 在世界面上的位置不影响结果，因为它们是零维顶点算符。[2] 注意 (30) 式中关联函数的 $'$ ；在开-闭弦场论中，我们需要定义更广义的内积，形式类比 (12) 式，在右侧带有常数项 K 。

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[2] The difference in sign between (11) and (30) was chosen so that we can use a uniform sign for the string field theory action in different theories. Indeed, consider a physical -1 picture vertex operator $e^{-\phi}V$ where V is a GSO odd, dimension 1/2 matter sector primary, normalized as $V(z)V(w) \simeq (z-w)^{-1}$. Then $e^{-\phi}(z)V(z)e^{-\phi}(w)V(w) \simeq -(z-w)^{-2}e^{-2\phi}(w)$, with a minus sign that is compensated by changing the sign of the overlap.

[2] (11) 式和 (30) 式的符号差异是为了让不同理论中的弦场论作用量采用统一的符号。具体来说，考虑一个-1 鬼图的物理顶点算符 $e^{-\phi}V$ ，其中 V 是 GSO 奇、维数为 1/2 的物质层主算符，归一化为 $V(z)V(w) \simeq (z-w)^{-1}$ 。此时 $e^{-\phi}(z)V(z)e^{-\phi}(w)V(w) \simeq -(z-w)^{-2}e^{-2\phi}(w)$ 会出现一个负号，该负号可以通过改变重叠积分的符号抵消。

For heterotic string theory, the left chiral (anti-holomorphic) sector is like the bosonic string theory, while the right chiral sector is like the type II string theory. In this theory, there is only one GSO operator, and we include in the spectrum only those states that are invariant under this GSO operator. The vacuum is normalized as

对于杂弦理论，左手征 (反全纯) 区与玻色弦类似，右手征区与 II 型弦类似。该理论中仅存在一个 GSO 算符，我们仅将该 GSO 算符下不变的态纳入谱中。真空归一化为:

$$\langle k;0|c_{-1}c_0c_1e^{-2\phi}|k';0\rangle = (2\pi)^D\delta^{(D)}(k+k'), \tag{31}$$

with the difference in sign relative to (29) having the same origin as described in footnote 2.

与 (29) 式的符号差异，其来源和脚注 2 中描述的一致。

In the construction of the interaction terms in string field theory, correlation functions of local operators on various Riemann surfaces will play an important role. These will depend on the moduli of the Riemann surfaces and the locations of the vertex operators; these insertion points are called ”punctures” of the surface. We shall collectively call these the moduli of punctured Riemann surfaces. Since general vertex operators are not invariant under arbitrary conformal transformations, these correlation functions also depend on a choice of a local coordinate system w_i in which the i -th vertex operator V_i is inserted on the Riemann surface at $w_i = 0$. On a Riemann surface, no local coordinate system is preferred, so a choice must be made of a local coordinate at each puncture. If z denotes a coordinate system on the Riemann surface $\sum$ and if z is related to w_i via $z = f_i(w_i)$ near $w_i = 0$, then the correlation function $\mathcal{C}$ may be written as

在弦场论相互作用项的构造中，定义在不同黎曼面上的局部算符的关联函数发挥着重要作用。这些关联函数依赖于黎曼面的模和顶点算符的位置；这些插入点被称为曲面的“puncture(穿刺点)”。我们将它们统称为带穿刺黎曼面的模。由于一般的顶点算符在任意共形变换下不是不变量，这些关联函数还依赖于局部坐标系的选择 w_i ，第 i 个顶点算符 V_i 被插入黎曼面上 $w_i = 0$ 处的该坐标系中。黎曼面上本身没有优先的局部坐标系，因此必须为每个穿刺点选择一个局部坐标。若 z 表示黎曼面 $\sum$ 上的一个坐标系，且在 $w_i = 0$ 附近 z 通过 $z = f_i(w_i)$ 变换到 w_i ，那么关联函数 $\mathcal{C}$ 可以写为:

$$\mathcal{C} = \left\langle \prod_i f_i \circ V_i(0) \right\rangle_{\sum}, \tag{32}$$

where $f \circ V(w)$ denotes the conformal transform of V by the map f and the subscript $\sum$ indicates that the correlation function is being computed on the surface $\sum$. If V is a dimension $(\bar{h}, h)$ primary operator, then

其中 $f \circ V(w)$ 表示 V 经由映射 f 得到的共形变换，下标 $\sum$ 表示关联函数是在曲面 $\sum$ 上计算的。若 V 是维数为 $(\bar{h}, h)$ 的主算符，则:

$$f \circ V(w) = (f'(w))^h \left(\bar{f}'(\bar{w})\right)^{\bar{h}} V(f(w)). \tag{33}$$

For general operators, the transformation laws are more complicated. The object $f \circ V(w)$ is in fact $V(w)$ rewritten as an operator in the z plane using $z = f(w)$. Indeed, from the transformation of the primary V, one has

对于一般算符，变换规律会更复杂。实际上对象 $f \circ V(w)$ 就是利用 $z = f(w)$ 改写为 z 平面上的算符后的 $V(w)$ 。根据主算符 V 的变换性质，可以得到:

$$V(w)(dw)^h(d\bar{w})^{\bar{h}} = V(z)(dz)^h(d\bar{z})^{\bar{h}} \;\rightarrow\; V(w) = V(z(w))\left(\frac{dz}{dw}\right)^h \left(\frac{d\bar{z}}{dw}\right)^{\bar{h}}.$$

(34)

With $z = f(w)$, the above right-hand side coincides with the right-hand side of (33). This is why the correlation function $\mathcal{C}$ above is sometimes written as

代入 $z = f(w)$ 后，上述等式右侧与 (33) 式的右侧完全一致。这就是为什么上述关联函数 $\mathcal{C}$ 有时会被写为:

$$\mathcal{C} = \left\langle \prod_i V_i(w_i = 0) \right\rangle_{\sum}. \tag{35}$$

The expression (32) will be thought as an off-shell correlation function. An on-shell correlation function is one where the vertex operators are all conformal invariant, and therefore the conformal maps by the functions f_i have no effect.

我们将 (32) 式的表达式视为脱壳关联函数。在壳关联函数中所有顶点算符都是共形不变的，因此函数 f_i 给出的共形映射不会产生任何效应。

Throughout our analysis, we shall assume that the CFT correlation functions on Riemann surfaces, introduced above, follow the gluing axioms as described by G. Segal in [32]. For closed string punctures, this means the following. Suppose we have a pair of closed string punctures P_1 and P_2 , either on the same Riemann surface or on two different Riemann surfaces, and let w_1 and w_2 be the local coordinates at those punctures. Let $\{|\chi_r\rangle\}$ and $\{|\chi_r^c\rangle\}$ be a pair of complete set of basis states, satisfying

在整个分析过程中，我们假设上文引入的黎曼曲面上的 CFT 关联函数满足 G. Segal 在文献 [32] 中描述的粘合公理。对于闭弦穿孔，这意味着如下内容: 假设我们有一对闭弦穿孔 P_1 和 P_2 ，它们可以位于同一黎曼曲面上，也可以位于两个不同的黎曼曲面上，记 w_1 和 w_2 为这些穿孔处的局部坐标。令 $\{|\chi_r\rangle\}$ 和 $\{|\chi_r^c\rangle\}$ 为一组完备基态对，满足

$$\langle \chi_s^c \mid \chi_r \rangle = \delta_{rs} \;\Leftrightarrow\; \sum_r |\chi_r\rangle\langle \chi_r^c| = I, \tag{36}$$

where I is the identity operator. Let us suppose that we insert at the first puncture the state χ_r and at the second puncture the state χ_r^c , place χ_r to the extreme right and χ_r^c to the extreme left of the correlation function (picking up signs if needed), and sum over r . Then this is equivalent to computing the correlation function on a new Riemann surface without the punctures P_1 and P_2 and identifying the local coordinates w_1 and w_2 via the relation

其中 I 是单位算符。假设我们在第一个穿孔插入态 χ_r ， 在第二个穿孔插入态 χ_r^c ， 将 χ_r 放在关联函数的最右端，χ_r^c 放在关联函数的最左端 (必要时调整符号)，并对 r 求和。这等价于在去掉穿孔 P_1 和 P_2 的新黎曼曲面上计算关联函数，并通过如下关系等同局部坐标 w_1 和 w_2

$$w_1 w_2 = 1\,. \tag{37}$$

Inside the correlator, we preserve the original order of the rest of the operators and, in the case of gluing two different Riemann surfaces, place all the operators on the Riemann surface associated with the first puncture to the left of all the operators on the Riemann surface associated with the second puncture.

在关联函数内部，我们保留其余算符的原始顺序；对于粘合两个不同黎曼曲面的情况，将第一个穿孔所属黎曼曲面上的所有算符，放在第二个穿孔所属黎曼曲面上所有算符的左侧。

For open strings, we have a similar relation except that (37) is replaced by

对于开弦，我们有类似的关系，只是 (37) 被替换为

$$w_1 w_2 = -1 \tag{38}$$

to account for the difference in the BPZ conjugate operator in the two theories, i.e., $I_c(z) = 1/z$ and $I_o(z) = -1/z$.

这是为了适配两种理论中 BPZ 共轭算符的差异，即 $I_c(z) = 1/z$ 和 $I_o(z) = -1/z$ 。

For heterotic string and superstrings, the construction is similar with one important difference. When we glue two punctures on two different Riemann surfaces, the picture numbers of χ_r and χ_r^c are fixed by picture number conservation. However, when the two punctures lie on the same Riemann surface, the picture number $(p_r, \bar{p}_r)$ of χ_r can be arbitrary, and that of χ_r^c is fixed to be $(2-p_r, 2-\bar{p}_r)$. In this case, instead of summing over all picture number states, we only sum over states carrying a fixed picture number. The canonical choice is picture number -1 in the NS sector and $-1/2$ for χ_r and $-3/2$ for χ_r^c (or vice versa) for the R sector, since only in these sectors, the L_0 eigenvalue is bounded from below. In contrast, both in the bosonic and the superstring theories, we sum over χ_r of all ghost numbers.

对于杂化弦和超弦，构造过程类似，但存在一个重要区别: 当我们粘合两个不同黎曼曲面上的两个穿孔时，χ_r 和 χ_r^c 的图数由图数守恒固定。但当两个穿孔位于同一黎曼曲面上时，χ_r 的图数 $(p_r, \bar{p}_r)$ 可以任意，而 χ_r^c 的图数固定为 $(2-p_r, 2-\bar{p}_r)$ 。这种情况下，我们不对所有图数的态求和，仅对携带固定图数的态求和。标准选择是:NS 扇区图数为-1，R 扇区 χ_r 为 $-1/2$ 、χ_r^c 为 $-3/2$ (或反之)，因为只有在这些扇区中，L_0 的本征值才有下界。相比之下，无论是玻色弦还是超弦理论，我们都会对所有鬼数的 χ_r 求和。

Bosonic String Amplitudes and Their Off-Shell Generalization

玻色弦振幅及其离壳推广

In this subsection, we shall describe the procedure for constructing amplitudes in closed bosonic string theory. The g loop n -point amplitude of BRST invariant external states with vertex operators $V_1, \cdots, V_n$ may be expressed as an integral of an appropriate correlation function of these vertex operators over the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n punctures. The real dimension of this moduli space is:

在本小节中，我们将描述闭玻色弦理论中构造振幅的过程。BRST 不变外态 (带有顶点算子 $V_1, \cdots, V_n$) 的 g 圈 n 点振幅可以表示为: 这些顶点算子的适当关联函数在模空间 $\mathcal{M}_{g,n}$ 上的积分，该模空间对应亏格为 g 、带有 n 个孔的黎曼曲面。该模空间的实维数为:

$$d_{g,n} \equiv \dim_{\mathbb{R}}\left(\mathcal{M}_{g,n}\right) = 6g - 6 + 2n. \tag{39}$$

Moreover, we also note that the Euler number $\chi_{g,n}$ of the surfaces in this moduli space is:

此外，我们还指出，该模空间中曲面的欧拉数 $\chi_{g,n}$ 为:

$$\chi_{g,n} = 2 - 2g - n \tag{40}$$

If we take generic representatives of the BRST cohomology that are not dimension zero primary fields, then the correlation function depends not only on the moduli of the Riemann surface but also on the choice of local coordinates at the punctures. Therefore, a precise description of the string amplitude requires us to go beyond the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces [33].

如果我们选取 BRST 上同调的非零维基本场的一般代表元，那么关联函数不仅依赖于黎曼曲面的模，还依赖于孔处局部坐标的选取。因此，对弦振幅的精确描述要求我们跳出黎曼曲面的模空间 $\mathcal{M}_{g,n}$ 进行讨论 [33]。

Let $\widehat{\mathcal{P}}_{g,n}$ be a fiber bundle whose base is the moduli space $\mathcal{M}_{g,n}$ and whose fiber gives the choice of local coordinates at the punctures up to phases. On a punctured Riemann surface, a local coordinate w at a puncture can be described as an analytic map from a round disk $|w| \leq 1$ to a disk domain surrounding the puncture, with $w = 0$ mapping to the puncture. The map induces a parameterization of the boundary of the disk domain. By the Riemann mapping theorem, the coordinate map between the two disks is uniquely fixed by the (unparameterized) boundary of the disk domain if we state which is the point in the disk domain boundary that corresponds to $w = 1$. The position of this point on the disk domain is equivalent to specifying the phase of the local coordinate or the "origin" of the closed string. Thus, a local coordinate at a puncture, up to a phase, is determined simply by drawing an arbitrary disk domain surrounding the puncture.

设 $\widehat{\mathcal{P}}_{g,n}$ 为一个纤维丛，其底空间为模空间 $\mathcal{M}_{g,n}$ ，纤维对应相差相位的孔处局部坐标选取。在带孔黎曼曲面上，孔处的局部坐标 w 可以描述为从圆盘 $|w| \leq 1$ 到包围该孔的圆盘区域的解析映射，其中 $w = =0$ 映射到孔本身。该映射诱导出圆盘区域边界的参数化。根据黎曼映射定理，如果我们指定对应 $w = 1$ 的圆盘区域边界上的点，那么两个圆盘之间的坐标映射由 (未参数化的) 圆盘区域边界唯一确定。这个点在圆盘区域上的位置等价于指定局部坐标的相位，也就是闭弦的“原点”。因此，相差一个整体相位的孔处局部坐标，仅需画出包围该孔的任意圆盘区域即可确定。

In the $\widehat{\mathcal{P}}_{g,n}$ bundle, the projection operator down to the base consists in forgetting the local coordinates. The phase ambiguity of the local coordinates in $\widehat{\mathcal{P}}_{g,n}$, happily, does not affect the correlation functions of states due to the conditions (13). There are sections $\mathcal{S}_{g,n}$ of this bundle, thanks to the phase ambiguity, there would be no sections if the fiber contained information on the phase of the local coordinates. The inability to fix the phase of the local coordinates continuously over the moduli space is the geometrical basis for the condition $L_0^- = 0$ satisfied by the off-shell states. The related $b_0^- = 0$ condition is needed for the construction of differential forms to be integrated over moduli space.

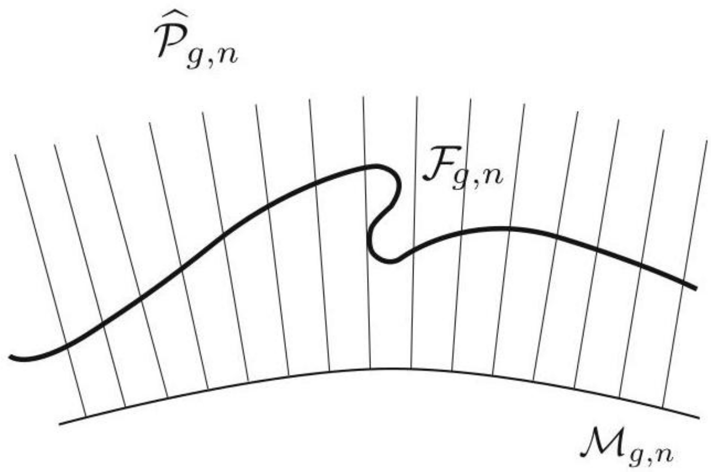
在 $\widehat{\mathcal{P}}_{g,n}$ 丛中，投影到底空间的操作就是忽略局部坐标。幸运的是，由于条件 (13) 的存在，$\widehat{\mathcal{P}}_{g,n}$ 中局部坐标的相位歧义不会影响关联函数。该丛存在截面 $\mathcal{S}_{g,n}$ ——这正是相位歧义带来的结果: 如果纤维包含局部坐标的相位信息，该丛就不存在截面。无法在整个模空间上连续固定局部坐标的相位，就是离壳态满足条件 $L_0^- = 0$ 的几何基础。相关的 $b_0^- = 0$ 条件是构造可在模空间上积分的微分形式所必需的。

In bosonic string theory, amplitudes can be computed using sections, but more flexibility is useful, keeping in mind the generalization of this construction to superstrings. We can use a subspace $\mathcal{F}_{g,n}$ of the bundle such that the projection map $\mathcal{F}_{g,n} \to \mathcal{M}_{g,n}$ is a map of degree one. [3] This means that a generic surface is counted once with multiplicity. The situation is sketched in Fig. 1. It is worth considering even more general situations where $\mathcal{F}_{g,n}$ is not a submanifold but rather a singular chain describing an element of the homology group $H_{6g-6+2n}\left(\widehat{\mathcal{P}}_{g,n}\right)$. This takes care of situations where $\mathcal{F}_{g,n}$ could have self-intersections or may be a formal weighted sum of disconnected spaces [34]. The replacement of subspaces for chains is no problem for the construction of amplitudes because forms are naturally integrated over chains. In this more general case, we will simply say that $\mathcal{F}_{g,n}$ is a chain (and so will be string vertices $\mathcal{V}_{g,n}$ when we discuss string field theory). The condition of a map of degree one, for the case of a chain $\mathcal{F}_{g,n}$, means that when pushed forward to $\mathcal{M}_{g,n}$, it represents the fundamental homology class of $\mathcal{M}_{g,n}$.

在玻色弦论中，我们可以利用截面计算振幅，但考虑到要将该构造推广到超弦，拥有更高的灵活性会更有帮助。我们可以取丛的一个子空间 $\mathcal{F}_{g,n}$ ，使得投影映射 $\mathcal{F}_{g,n} \to \mathcal{M}_{g,n}$ 是一次一映射。[3] 这意味着一般曲面都会被计数一次，重数为 1。该情形的示意图见图 1。我们还值得考虑更一般的情况: 此时 $\mathcal{F}_{g,n}$ 不是子流形，而是描述同调群 $H_{6g-6+2n}\left(\widehat{\mathcal{P}}_{g,n}\right)$ 中一个元素的奇异链。这就可以处理 $\mathcal{F}_{g,n}$ 可能存在自交点，或是若干不连通空间的形式加权和这类情况 [34]。对于振幅构造而言，将子空间替换为链不存在问题，因为形式天然就可以在链上积分。在这个更一般的情形下，我们就直接称 $\mathcal{F}_{g,n}$ 为一个链 (我们讨论弦场论时，弦顶点 $\mathcal{V}_{g,n}$ 也同样是链)。对于 $\mathcal{F}_{g,n}$ 是链的情况，一次一映射的条件意味着，当它被推前到 $\mathcal{M}_{g,n}$ 时，它表示 $\mathcal{M}_{g,n}$ 的基本同调类。

Fig. 1 A subspace $\mathcal{F}_{g,n}$ of $\widehat{\mathcal{P}}_{g,n}$ for which the map to the base $\mathcal{M}_{g,n}$ is a degree one map is all that is required to compute string amplitudes. The space $\mathcal{F}_{g,n}$, as shown, need not be a section of this bundle

图 1 要计算弦振幅，仅需满足: $\widehat{\mathcal{P}}_{g,n}$ 存在子空间 $\mathcal{F}_{g,n}$ ，且到基空间 $\mathcal{M}_{g,n}$ 的映射是一次一映射。如图所示，$\mathcal{F}_{g,n}$ 不需要是该丛的一个截面



The g -loop, n -point amplitude $\mathcal{A}_g(V_1, \cdots, V_n)$ of the BRST invariant operators is now given by

BRST 不变算符的 g 圈 n 点振幅 $\mathcal{A}_g(V_1, \cdots, V_n)$ 现在可写为

$$\mathcal{A}_g(V_1, \cdots, V_n) = (g_s)^{-\chi_{g,n}} \int_{\mathcal{F}_{g,n}} \Omega^{(g,n)}_{6g-6+2n}(V_1, \cdots, V_n), \tag{41}$$

where g_s is the string coupling constant and $\Omega^{(g,n)}_p(V_1, \cdots, V_n)$ denotes a properly normalized p -form on $\widehat{\mathcal{P}}_{g,n}$. The form in the above amplitude is of degree $d_{g,n}$. We will now explain how, more generally, we define the form $\Omega^{(g,n)}_p(A_1, \cdots, A_n)$ with A_i arbitrary vertex operators in $\mathcal{H}_c$.

其中 g_s 为弦耦合常数，$\Omega^{(g,n)}_p(V_1, \cdots, V_n)$ 表示定义在 $\widehat{\mathcal{P}}_{g,n}$ 上的归一化 p -形式。上述振幅中的形式次数为 $d_{g,n}$ 。下文我们将推广说明，如何在 A_i 顶点算符任意取定的 $\mathcal{H}_c$ 中定义形式 $\Omega^{(g,n)}_p(A_1, \cdots, A_n)$ 。

Given a Riemann surface $\sum_{g,n}$ of genus g carrying n punctures, we first decompose it into a collection of n disjoint disks $D_1, \cdots, D_n$, one around each puncture, and $(2g-2+n)$ spheres $S_1, \cdots S_{2g-2+2n}$, each with three holes. The $3(2g-2+n)$ boundaries of the spheres and the n boundaries of the disks, which we shall call gluing circles, are glued pairwise at $(3g-3+2n)$ circles $C_1, \cdots, C_{3g-3+2n}$. An example of such a decomposition has been shown in Fig. 2. We choose some complex coordinate system w_a on the disk D_a such that the a -th puncture is situated at $w_a = 0$. These are the local coordinates at the punctures (but note that the boundary of the disk is not necessarily the locus of $|w_a| = 1$). We also choose some complex coordinate system z_i on the sphere S_i . Here $a = 1, \ldots, n$ and $i = 1, \ldots, 2g-2+n$.

给定亏格为 g 、带有 n 个孔的黎曼曲面 $\sum_{g,n}$ ，我们首先将其分解为 n 个互不相交的圆盘 $D_1, \cdots, D_n$ (每个圆盘围绕一个孔) 与 $(2g-2+n)$ 个三孔球面 $S_1, \cdots S_{2g-2+2n}$ 。这些球面的 $3(2g-2+n)$ 个边界和圆盘的 n 个边界被称为粘合圆，它们在 $(3g-3+2n)$ 个圆 $C_1, \cdots, C_{3g-3+2n}$ 处成对粘合。图 2 已给出这类分解的一个示例。我们在圆盘 D_a 上选取某个复坐标系 w_a ，使得第 a 个孔位于 $w_a = 0$ 。这些是孔处的局部坐标 (但注意圆盘边界不一定是 $|w_a| = 1$ 所在的轨迹)。我们还在球面 S_i 上选取某个复坐标系 z_i ，此处为 $a = 1, \ldots, n$ 和 $i = 1, \ldots, 2g-2+n$ 。

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[3] The degree of a continuous mapping between two compact oriented manifolds of the same dimension is an integer that represents the number of times the domain manifold wraps around the range manifold under the map. For any regular point x on the range manifold, the preimage in the domain manifold is a set of points $\{x_1, x_2, \cdots, x_n\}$, such that the map from a neighborhood of each x_i to a neighborhood of x is a diffeomorphism. The degree of the map is the number x_i ’ s for which the map is orientation preserving minus the number of x_j ’ s for which the map is orientation reversing. That degree must be independent of the point x on the range manifold. Any map of degree different from zero must be surjective. An example of a degree one map that is not a section has been illustrated in Fig. 1.

[3] 两个相同维度紧致定向流形之间连续映射的度是一个整数，表示原流形在映射下环绕像流形的次数。对于像流形上的任意正则点 x ，它在原流形中的原像是点集 $\{x_1, x_2, \cdots, x_n\}$ ，从每个 x_i 的邻域到 x 的邻域的映射都是微分同胚。映射的度等于保定向映射的 x_i 数量减去逆定向映射的 x_j 数量。该度一定与像流形上点 x 的选取无关。任何度非零的映射都必是满射。图 1 给出了一个不是截面的一次度映射示例。

We denote by σ_s and τ_s the complex coordinate system on the left and right of the gluing circle C_s , respectively, for some given choice of orientation of C_s . There are two possible gluing patterns: disk to sphere, where one out of (σ_s, τ_s) is a w_a and the other is a z_i , and sphere to sphere, where one out of (σ_s, τ_s) is a z_i and the other a z_j , with i and j different or the same. The moduli of the Riemann surface $\sum_{g,n}$ and the local coordinate system at the punctures are fixed by the transition functions relating σ_s and τ_s . Two sets of transition functions are declared equivalent if they can be related by reparametrization of w_a keeping the puncture at $w_a = 0$ fixed and/or reparametrization of the coordinates z_i . The space of inequivalent transition functions gives the moduli space $\mathcal{M}_{g,n}$. On the other hand, the bigger space $\widehat{\mathcal{P}}_{g,n}$ arises when two sets of transition functions are declared equivalent if they can be related by reparametrization of the coordinates z_i keeping the w_a ’ s fixed up to phase rotations. Therefore, if u stands for the set of coordinates of $\widehat{\mathcal{P}}_{g,n}$, we can write the transition functions as follows:

对于粘合圆 C_s 给定的某种定向，我们分别用 σ_s 和 τ_s 表示粘合圆左侧和右侧的复坐标系。粘合共有两种可能模式: 盘粘球面，其中 (σ_s,τ_s) 中的一个是 w_a ，另一个是 z_i ；以及球面粘球面，其中 (σ_s,τ_s) 中的一个是 z_i ，另一个是 z_j ，此时 i 与 j 可相同也可不同。黎曼曲面 $\sum_{g,n}$ 的模和孔处的局部坐标系由联系 σ_s 和 τ_s 的转移函数确定。若两组转移函数可通过保持孔在 $w_a=0$ 固定的 w_a 重新参数化，或 (和) 坐标 z_i 的重新参数化建立联系，则称二者等价。不等价转移函数的空间构成模空间 $\mathcal{M}_{g,n}$ 。另一方面，若仅要求两组转移函数可通过保持 w_a 差一个相位旋转固定的坐标 z_i 重新参数化建立联系就称二者等价，就得到更大的空间 $\widehat{\mathcal{P}}_{g,n}$ 。因此，若 u 表示 $\widehat{\mathcal{P}}_{g,n}$ 的坐标集合，我们可将转移函数写作如下形式:

$$\sigma_s = F_s(\tau_s, u). \tag{42}$$

For a given tangent vector $\partial/\partial u^i$ of $\widehat{\mathcal{P}}_{g,n}$, we introduce the world-sheet operator $\mathcal{B}\left[\frac{\partial}{\partial u^i}\right]$ defined as a sum of contour integrals along all the gluing circles C_s :

对于 $\widehat{\mathcal{P}}_{g,n}$ 的给定切向量 $\partial/\partial u^i$ ，我们引入世界面算符 $\mathcal{B}\left[\frac{\partial}{\partial u^i}\right]$ ，其定义为所有粘合圆 C_s 上的围道积分之和:

$$\mathcal{B}\left[\frac{\partial}{\partial u^i}\right] \equiv \sum_s \left[\oint_{C_s} \frac{\partial F_s}{\partial u^i} d\sigma_s b(\sigma_s) + \oint_{C_s} \frac{\partial \bar{F}_s}{\partial u^i} d\bar{\sigma}_s \bar{b}(\bar{\sigma}_s)\right], \tag{43}$$

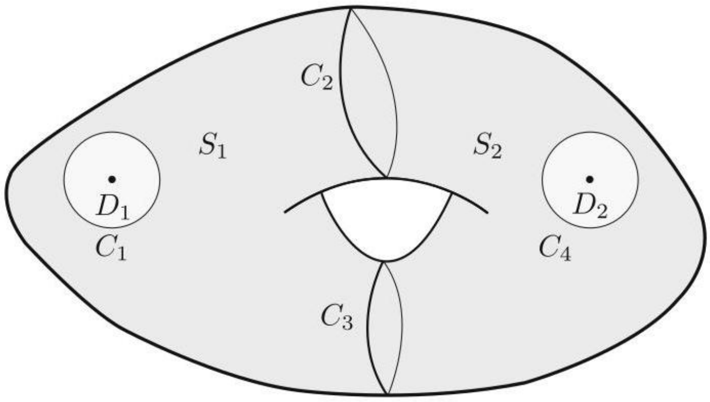
where $\oint$ is normalized as in (3) and, as mentioned above, the orientation of C_s is such that σ_s is to the left of the contour and τ_s is to the right of the contour. The contraction of $\Omega_p^{(g,n)}(A_1,\cdots,A_n)$ with p tangent vectors of $\widehat{\mathcal{P}}_{g,n}$ is now defined as

其中 $\oint$ 按照式 (3) 的方式归一化，且如上所述，C_s 的定向满足: σ_s 位于围道左侧，τ_s 位于围道右侧。$\Omega_p^{(g,n)}(A_1,\cdots,A_n)$ 与 $\widehat{\mathcal{P}}_{g,n}$ 的 p 个切向量的缩并现在定义为

$$\begin{aligned}&\Omega_p^{(g,n)}(A_1,\cdots,A_n)\left[\frac{\partial}{\partial u^{j_1}},\cdots,\frac{\partial}{\partial u^{j_p}}\right]\\ &\equiv \left(-\frac{1}{2\pi i}\right)^{3g-3+n}\left\langle \mathcal{B}\left[\frac{\partial}{\partial u^{j_1}}\right]\cdots\mathcal{B}\left[\frac{\partial}{\partial u^{j_p}}\right]A_1\cdots A_n\right\rangle_{\Sigma_{g,n}},\end{aligned} \tag{44}$$

Fig. 2 Decomposition of a genus one surface with two punctures, denoted by dots, into two disks D_1 and D_2 and two three-holed spheres S_1 and S_2 , joined across four gluing circles C_1, C_2, C_3, C_4

图 2 带两个孔 (以点标记) 的亏格 1 曲面分解为两个盘 D_1 、D_2 和两个三孔球面 S_1 、S_2 ，各部分通过四个粘合圆 C_1, C_2, C_3, C_4 拼接在一起



where $\langle\rangle_{\sum_{g,n}}$ denotes correlation function on the n -punctured Riemann surface $\sum_{g,n}$ and the vertex operators $A_1, \cdots A_n$ are inserted at the punctures $w_1 = 0, \cdots, w_n = 0$ using the local coordinates $w_1, \cdots, w_n$ (see (35)). As anticipated, we shall take (44) to be the definition of $\Omega_p^{(g,n)}(A_1, \cdots, A_n)$ for any $A_i \in \mathcal{H}_c$, not necessarily just for BRST invariant A_i 's. The origin of the normalization prefactor $\left(-\frac{1}{2\pi i}\right)^{3g-3+n}$ chosen above will be explained in section "World-Sheet String Amplitudes from String Field Theory" (see (234)). In writing (44), we have assumed that the orientation of the integration measure in (41) is the canonical one inherited from the complex structure in $\mathcal{M}_{g,n}$, i.e., if $u = u_x + iu_y$ is a complex coordinate the $du \wedge d\bar{u} = -2idu_x \wedge du_y$ with the integral of $du_x \wedge du_y$ yielding a positive result.

其中 $\langle\rangle_{\sum_{g,n}}$ 表示 n 个穿孔黎曼曲面 $\sum_{g,n}$ 上的关联函数，顶点算子 $A_1, \cdots A_n$ 利用局部坐标 $w_1, \cdots, w_n$ 插入在穿孔 $w_1 = 0, \cdots, w_n = {}_0$ 处 (见式 (35))。正如预期，我们将 (44) 作为任意 $A_i \in \mathcal{H}_c$ 情况下 $\Omega_p^{(g,n)}(A_1, \cdots, A_n)$ 的定义，并不局限于 BRST 不变的 A_i 。上文所选归一化前置因子 $\left(-\frac{1}{2\pi i}\right)^{3g-3+n}$ 的来源将在“弦场论导出的世界面弦振幅”一节中说明 (见式 (234))。在写下 (44) 时我们假设，(41) 中积分测度的取向是继承自 $\mathcal{M}_{g,n}$ 复结构的标准取向，即若 $u = u_x + iu_y$ 是复坐标，则对 $du_x \wedge du_y$ 积分后 $du \wedge d\bar{u} = -2idu_x \wedge du_y$ 得到正结果。

The above formalism, where antighost insertions can be supported on all gluing circles, represents an extension of the Schiffer variation formalism used in the operator formulation of CFT's [33, 35, 36] as well as in formulation of closed string field theory and a number of computations in this theory [6, 37]. As befits the operator formulation of CFT, antighost insertions are all supported only on the gluing circles that surround the punctures where the external states are inserted. All tangents on $\widehat{\mathcal{P}}_{g,n}$ can be represented by deformations of the gluing relations on those circles [38]. Thus, all the u moduli appear in transition functions relating w_a and z_i . The Schiffer vector v_a^u , defined on a neighborhood of the gluing circle for the $w_a = 0$ puncture, enters the antighost insertion in the form $\oint dw_a b(w_a) v_a^u(w)$, plus antiholomorphic parts.

上述形式体系允许反鬼插入支撑在所有粘合圆上，是希弗变分形式体系的推广: 希弗变分已用于共形场论的算子表述 [33, 35, 36]，也用于闭弦场论的构造及该理论中的诸多计算 [6, 37]。在符合共形场论算子表述的传统形式中，反鬼插入仅支撑在环绕外部态插入穿孔的粘合圆上。$\widehat{\mathcal{P}}_{g,n}$ 上的所有切向量都可以通过这些圆上粘合关系的变形表示 [38]。因此，所有 u 模都出现在关联 w_a 与 z_i 的转移函数中。定义在第 $w_a = 0$ 个穿孔粘合圆邻域上的希弗向量 v_a^u 以 $\oint dw_a b(w_a) v_a^u(w)$ 加反全纯部分的形式进入反鬼插入。

It is instructive to see how our prescription converts an unintegrated vertex operator $\bar{c}cV$, with V a dimension $(1,1)$ primary operator in the matter sector, to a two-form operator $dy \wedge d\bar{y}V(y)$ that can be integrated. For this, let w be the local coordinate at the puncture situated at $w = 0$ and z be the coordinate system in a patch of the surface that encloses the open set covered by the w coordinate. On the curve separating the patches labelled by z and w coordinates, let z and w be related via

我们可以通过一个实例清晰说明本文的规则如何将未积分顶点算子 $\bar{c}cV$ (其中 V 是物质部分维度为 $(1,1)$ 的初态算子) 转化为可积分的二元算子 $dy \wedge d\bar{y}V(y)$。为此，设 w 是位于 $w=0$ 处穿孔的局部坐标，z 是曲面包片中覆盖 w 坐标所张成开集的坐标系。在分隔 z 坐标标号片区与 w 坐标标号片区的曲线上，z 与 w 满足下述关系:

$$z = F(w,u), \tag{45}$$

where $u = (u^1, u^2)$ are a pair of real coordinates that label the location of the puncture in the z coordinate. We shall assume that $F(w,u)$ is analytic in the patch covered by the w coordinate system, so that it can be used to extend the z coordinate into this patch. In that case, if we define

其中 $u = (u^1, u^2)$ 是一对实坐标，用来标记穿孔在 z 坐标系中的位置。我们假设 $F(w,u)$ 在 w 坐标系覆盖的片区中解析，因此可利用 $F(w,u)$ 将 z 坐标延拓到该片区。这种情况下，若我们定义

$$y(u) = F(0,u), \tag{46}$$

then y can be viewed as the location of the puncture in the z coordinate system, extended by (45) into the full patch covered by the w coordinate system. Comparing with (42), we see that z plays the role of σ_s and w plays the role of τ_s . Thus, up to additional insertions (denoted by dots) that we do not focus on, the effect of inserting the vertex operator $c\bar{c}V$ is represented by the two-form

于是 y 可视为打孔点在 z 坐标系中的位置，该坐标系通过式 (45) 扩展到 w 坐标系覆盖的整个区域。与式 (42) 对比可知，z 对应 σ_s 的角色，w 对应 τ_s 的角色。因此，暂不讨论额外插入项 (用点表示)，插入顶点算子 $c\bar{c}V$ 的效果可用下述二形式表示

$$\Omega_2(c\bar{c}V) = \left(-\frac{1}{2\pi i}\right) du^1 \wedge du^2 \left\langle \cdots \mathcal{B}\left[\frac{\partial}{\partial u^1}\right] \mathcal{B}\left[\frac{\partial}{\partial u^2}\right] c\bar{c}V(w=0) \right\rangle, \tag{47}$$

where it is understood that $V(w)$ means that V is inserted using the w coordinate system. We have included the extra factor of $-1/(2\pi i)$, consistent with (44), having increased the number of punctures by one. Using (43), we now get

此处约定 $V(w)$ 表示 V 是通过 w 坐标系插入的。由于打孔数增加了 1，我们按式 (44) 添加了额外因子 $-1/(2\pi i)$ ，代入式 (43) 后可得

$$\mathcal{B}\left[\frac{\partial}{\partial u^1}\right] \mathcal{B}\left[\frac{\partial}{\partial u^2}\right] = \left[\oint b(z)\, dz \frac{\partial F(w,u)}{\partial u^1} + \oint \bar{b}(\bar{z})\, d\bar{z} \frac{\partial \overline{F(w,u)}}{\partial u^1}\right]$$

(48)

$$\left[\oint b(z)\, dz \frac{\partial F(w,u)}{\partial u^2} + \oint \bar{b}(\bar{z})\, d\bar{z} \frac{\partial \overline{F(w,u)}}{\partial u^2}\right],$$

where the contours are along the boundary separating the patches covered by the z and w coordinate systems, i.e., surrounding the insertion at $w = 0$. Each of the contours runs clockwise so that they keep the z coordinate system to the left, but since we have product of two such contours, we can take them to be anti-clockwise. Since $c\bar{c}V$ is a dimension zero primary, we can express $c\bar{c}V(w = 0$ as $c\bar{c}V(z = y$. We carry out the integration over $z, \bar{z}$ by deforming the contours toward $z = y$ and picking up residues from the operator products $b(z)\, c\bar{c}V\,(z = y)$ and $\bar{b}\,(\bar{z})\, c\bar{c}V(z = \ y)$. After using $y = F(0, u)$, we get

其中积分围道沿 z 与 w 坐标系覆盖区域的分界线，也就是环绕 $w = 0$ 处的插入点。每条围道都沿顺时针方向，这样就能始终将 z 坐标系保持在左侧，但由于我们这里是两条围道的乘积，可以将其取为逆时针方向。由于 $c\bar{c}V$ 是零维基本场，我们可以将 $c\bar{c}V(w = 0$ 表示为 $c\bar{c}V(z = y$ 。我们对 $z, \bar{z}$ 积分时，将围道向 $z = y$ 变形，并从算符乘积 $b(z)\, c\bar{c}V\,(z = y)$ 和 $\bar{b}\,(\bar{z})\, c\bar{c}V(z = \ y)$ 中提取留数。代入 $y = F(0, u)$ 后，我们得到

$$\Omega_2\,(c\bar{c}V) = -\left(-\frac{1}{2\pi i}\right) du^1 \wedge du^2 \left(\frac{\partial y}{\partial u^1}\frac{\partial \bar{y}}{\partial u^2} - \frac{\partial y}{\partial u^2}\frac{\partial \bar{y}}{\partial u^1}\right)\langle \cdots V\,(z = y)\rangle.$$

(49)

Furthermore, we have

此外我们有

$$du^1 \wedge du^2 \left(\frac{\partial y}{\partial u^1}\frac{\partial \bar{y}}{\partial u^2} - \frac{\partial y}{\partial u^2}\frac{\partial \bar{y}}{\partial u^1}\right) = dy \wedge d\bar{y}. \tag{50}$$

Using this in (49), we thus find that

将此结果代入式 (49)，可得

$$\Omega_2\,(c\bar{c}V) = \frac{1}{2\pi i}\langle \cdots (dy \wedge d\bar{y} V\,(z = y))\rangle. \tag{51}$$

As expected, (44) builds a two-form ready for integration starting from an unintegrated vertex operator.

正如预期，式 (44) 从一个未积分顶点算子出发，构造出了可用于积分的二形式。

Using conformal field theory Ward identities, one can show that the action of the BRST operator on the states that enter the form $\Omega_p^{(g,n)}$ maps to an action of the exterior derivative on the form of one degree less [6,33]:

利用共形场论沃德恒等式可以证明，BRST 算子对进入形式 $\Omega_p^{(g,n)}$ 的态的作用，等价于外导数对低一次形式的作用 [6,33]:

$$\Omega_p^{(g,n)}\,(QA_1, A_2, \cdots, A_n) + \cdots + (-1)^{A_1 + \cdots + A_{n-1}}\,\Omega_p^{(g,n)}\,(A_1, A_2, \cdots, QA_n)$$

$$= (-1)^p\ \mathrm{d}\Omega_{p-1}^{(g,n)}\,(A_1, \cdots, A_n)\,, \tag{52}$$

where $(-1)^A$ is defined to be 1 if A has even ghost number and -1 if A has odd ghost number. Using this identity, one can formally show as follows that the amplitude is independent of the choice of the subspace $\mathcal{F}_{g,n}$

. Let $\mathcal{F}'_{g,n}$ be another subspace of $\widehat{\mathcal{P}}_{g,n}$, with $\mathcal{F}'_{g,n} \to \mathcal{M}_{g,n}$ also of degree one, and let $\mathcal{R}_{g,n}$ be any $6g-5+2n$ dimensional space that interpolates between $\mathcal{F}_{g,n}$ and $\mathcal{F}'_{g,n}$, so that $\partial\mathcal{R}_{g,n} = \mathcal{F}'_{g,n} - \mathcal{F}_{g,n}$. Then we have

其中当 A 鬼数为偶数时 $(-1)^A$ 定义为 1，当 A 鬼数为奇数时 $(-1)^A$ 定义为-1。利用该恒等式可以形式化证明振幅与子空间 $\mathcal{F}_{g,n}$ 的选取无关，证明过程如下: 设 $\mathcal{F}'_{g,n}$ 是 $\widehat{\mathcal{P}}_{g,n}$ 的另一个子空间，对应的 $\mathcal{F}'_{g,n} \to \mathcal{M}_{g,n}$ 同样是一次的，再设 $\mathcal{R}_{g,n}$ 是任意插值于 $\mathcal{F}_{g,n}$ 和 $\mathcal{F}'_{g,n}$ 之间的 $6g-5+2n$ 维空间，满足 $\partial\mathcal{R}_{g,n} = \mathcal{F}'_{g,n} - \mathcal{F}_{g,n}$ ，于是我们有

$$\int_{\mathcal{F}'_{g,n}} \Omega^{(g,n)}_{6g-6+2n}(V_1,\cdots,V_n) - \int_{\mathcal{F}_{g,n}} \Omega^{(g,n)}_{6g-6+2n}(V_1,\cdots,V_n)$$
$$= \int_{\mathcal{R}_{g,n}} \mathrm{d}\Omega^{(g,n)}_{6g-5+2n}(V_1,\cdots,V_n)\,. \tag{53}$$

Since $QV_i = 0$ for each i , the integrand on the right-hand side vanishes by (52), and thus the right-hand side vanishes.

由于对每个 i 都有 $QV_i = 0$ ，由式 (52) 可知右侧被积函数为零，因此右侧整体为零。

Equation (52) also can be used to give a formal proof that the BRST exact states have vanishing amplitude as long as all the other external states are BRST invariant. For this, let us suppose that $V_1 = QW_1$ and $V_2,\cdots,V_n$ are BRST invariant. Then using (52), we can write

式 (52) 还可用于形式化证明: 只要所有其他外态都是 BRST 不变的，BRST 恰当态的振幅就为零。为此，假设 $V_1 = QW_1$ 和 $V_2,\cdots,V_n$ 都是 BRST 不变的，利用式 (52) 我们可以写出

$$\Omega^{(g,n)}_{6g-6+2n}(QW_1,V_2,\cdots,V_n) = \mathrm{d}\Omega^{(g,n)}_{6g-7+2n}(W_1,V_2,\cdots,V_n)\,. \tag{54}$$

When we integrate this over $\mathcal{F}_{g,n}$, we get a vanishing result.

将其在 $\mathcal{F}_{g,n}$ 上积分后，结果为零。

Both these arguments are valid up to a subtlety with special "divisors," sets where the corresponding surfaces in the moduli space $\mathcal{M}_{g,n}$ degenerate. More precisely, one works with the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space, where there is no boundary, and the divisors are represented by nodal surfaces (see section "Geometric BV Master Equation and String Field Theory Master Equation".) The nodal surfaces are sets of real codimension two in the moduli space. The complication is that the integrand $\Omega^{(g,n)}$ for string amplitudes can diverge as we approach the divisors. One option to deal with this problem is to complexify the moduli space and deform the integration contour in this complexified moduli space such that the form $\Omega^{(g,n)}_{6g-7+2n}$ appearing on the right-hand side of (54) vanishes as we approach the end point of the integration region [39, 40] . This is equivalent to using the $i\varepsilon$ prescription in quantum field theory and works when the momentum flowing through the node is generic but fails when the momentum is forced to be on-shell due to conservation laws. As we shall describe, one of the applications of string field theory will be to make sense of these divergences and extract finite, unambiguous answers from them. This also allows us to evaluate the integral of (54) and show that it indeed vanishes.

这两种论证都成立，但存在一个涉及特殊“除子”的微妙问题——除子就是模空间中对应曲面 $\mathcal{M}_{g,n}$ 退化的集合。更准确地说，我们使用模空间的德利涅-芒福德紧化 $\overline{\mathcal{M}}_{g,n}$ ，该紧化没有边界，除子由结点曲面表示 (参见章节“几何 BV 主方程与弦场论主方程”)。结点曲面是模空间中实余维数为 2 的集合。问题的复杂之处在于，弦振幅的被积函数 $\Omega^{(g,n)}$ 在我们趋近除子时会发散。处理该问题的一种方案是将模空间复形，在复化后的模空间中对积分围道做形变，使得 (54) 右侧出现的形式 $\Omega^{(g,n)}_{6g-7+2n}$ 在我们趋近积分区域 [39, 40] 的端点时等于零。这等价于在量子场论中使用 $i\varepsilon$ 处方，当流过结点的动量是一般动量时该方法可行，但当动量受守恒约束必须在壳时它就失效了。正如我们将要说明的，弦场论的应用之一就是赋予这些发散意义，并从中抽取出有限、明确的结果。这也能让我们计算 (54) 的积分，证明它确实等于零。

Amplitudes in Closed Superstring Theories

闭超弦理论中的振幅

The amplitudes in superstring theory are defined similarly, but with a few differences. First of all, each physical state has infinite number of representatives, one in each integer picture number for NS sector states and one in each integer + half picture number for R sector states. We choose picture -1 to represent NS sector states and picture $-1/2$ to represent R sector states. If we denote by $\mathcal{H}_c$ the space of closed string states carrying these picture numbers and satisfying the subsidiary conditions (13), then the physical closed string states are elements of the BRST cohomology in $\mathcal{H}_c$ carrying ghost number two.

超弦理论中的振幅定义类似，但存在几处差异。首先，每个物理态都有无穷多个代表元:NS sector 态每个整数鬼图编号各有一个，R sector 态每个半整数鬼图编号各有一个。我们选取鬼图-1 代表 NS sector 态，选取鬼图 $-1/2$ 代表 R sector 态。若将携带这些鬼图编号且满足辅助条件 (13) 的闭弦态空间记为 $\mathcal{H}_c$ ，那么物理闭弦态就是 $\mathcal{H}_c$ 中鬼数为 2 的 BRST 上同调元。

Consider now a genus g correlation function. This correlation function vanishes unless the total picture number of all the operators is $(2g-2)$, separately in the holomorphic and the anti-holomorphic sectors. Hence, we need to add some picture changing operators on the Riemann surfaces. Let $n_{p,q}$ be the number of vertex operators A_i with picture number (p,q) . Given the picture number assignment for the separate NS and R sectors, the possible n ’ s are:

现在考虑亏格 g 关联函数。该关联函数只有当全纯区和反全纯区所有算符的总鬼图编号分别为 $(2g-2)$ 时才非零。因此我们需要在黎曼面上添加若干鬼图变换算符。设 $n_{p,q}$ 是鬼图编号为 (p,q) 的顶点算符 A_i 的数量。给定 NS sector 和 R sector 各自的鬼图编号分配后，可能的 n 为:

$$n_{-1,-1},\ n_{-1,-1/2},\ n_{-1/2,-1},\ n_{-1/2,-1/2}. \tag{55}$$

The numbers N_L and N_R of PCOs that we need on the left and the right sectors, respectively, are obtained from the sum rules

我们分别在左区和右区需要的鬼图变换算符 (PCO) 数量 N_L 和 N_R 可由求和规则得到

$$2g-2=N_L-n_{-1,-1}-n_{-1,-1/2}-\frac{1}{2}n_{-1/2,-1}-\frac{1}{2}n_{-1/2,-1/2},$$

$$2g-2=N_R-n_{-1,-1}-n_{-1/2,-1}-\frac{1}{2}n_{-1,-1/2}-\frac{1}{2}n_{-1/2,-1/2}. \tag{56}$$

Let $y_1,\cdots,y_{N_R}$ and $\bar{y}_1,\cdots,\bar{y}_{N_L}$, collectively denoted as $y_\alpha,\bar{y}_\alpha$, be the positions where we will insert right and left sector operators that change picture number. The operators to be inserted will be either $(-\partial\xi)$ or the standard PCO $\mathcal{X}_0$ for the holomorphic sector (right) and either $\left(-\overline{\partial\xi}\right)$ or the standard PCO $\overline{\mathcal{X}}_0$ for the anti-holomorphic sector (left). The coordinates $y_\alpha,\bar{y}_\alpha$ are taken to be in the w_a or z_i coordinate system depending on whether they lie on the disk D_a or the sphere S_i . No local coordinate is needed at the location of the PCO insertions because these are primary fields of dimension zero. No local coordinate will be needed for the $(-\partial\xi)$ and $\left(-\overline{\partial\xi}\right)$ insertions because they are dimension one but appear as invariant one-forms.

设 $y_1,\cdots,y_{N_R}$ 和 $\bar{y}_1,\cdots,\bar{y}_{N_L}$ 是我们插入右区和左区鬼图变换算符的位置，二者统一记为 $y_\alpha,\bar{y}_\alpha$。对于全纯区 (右区)，待插入算符为 $(-\partial\xi)$ 或标准鬼图变换算符 $\mathcal{X}_0$；对于反全纯区 (左区)，待插入算符为 $\left(-\overline{\partial\xi}\right)$ 或标准鬼图变换算符 $\overline{\mathcal{X}}_0$。坐标 $y_\alpha,\bar{y}_\alpha$ 根据其位于圆盘 D_a 还是球面 S_i， 分别采用 w_a 或 z_i 坐标系。鬼图变换算符插入位置不需要局部坐标，因为这些算符都是维度为 0 的基本场。$(-\partial\xi)$ 和 $\left(-\overline{\partial\xi}\right)$ 插入也不需要局部坐标，因为它们维度为 1，但以不变 1-形式的形式出现。

We regard these positions $y_\alpha,\bar{y}_\alpha$ as extra fiber data to be added to $\widehat{\mathcal{P}}_{g,n}$ to form a new bundle $\widehat{\mathcal{P}}^s_{g,n}$, with $s=N_L+N_R$. Now a point of $\widehat{\mathcal{P}}^s_{g,n}$ is a Riemann surface $\sum_{g,n}$ with local coordinates (up to phases) at the punctures and a choice of s positions for the insertion of operators that change picture number. [4] Therefore, $\widehat{\mathcal{P}}^s_{g,n}$, in addition to the familiar bosonic moduli tangent vectors $\partial/\partial u^i$, now has tangent vectors $\partial/\partial y_\alpha$ and $\partial/\partial\bar{y}_\beta$ associated with changing the position of the operators that change picture number. A p form $\Omega^{(g,n)}_p$ acts on k moduli vectors, ℓ tangent vectors for right sector picture changing insertions and $\overline{\ell}$ tangent vector for left sector picture changing insertions, with

我们将这些位置 $y_\alpha,\bar{y}_\alpha$ 视为额外纤维数据，添加到 $\widehat{\mathcal{P}}_{g,n}$ 中以构成新丛 $\widehat{\mathcal{P}}^s_{g,n}$， 满足 $s=N_L+N_R$。此时 $\widehat{\mathcal{P}}^s_{g,n}$ 的一个点是黎曼曲面 $\sum_{g,n}$， 它带有孔点处的局部坐标 (相差相位范围内)，并且为改变图数的算符插入选定了 s 个位置。[4] 因此，除了常见的玻色模切向量 $\partial/\partial u^i$ 外，$\widehat{\mathcal{P}}^s_{g,n}$ 现在还拥有切向量 $\partial/\partial y_\alpha$ 和 $\partial/\partial\bar{y}_\beta$， 这些切向量对应改变图数变换算符的位置。一个 p 型的形式 $\Omega^{(g,n)}_p$ 作用于 k 个模向量、ℓ 个右区图变换插入切向量和 $\overline{\ell}$ 个左区图变换插入切向量，有

$$p=k+\ell+\overline{\ell}. \tag{57}$$

We define

我们定义

$$\Omega^{(g,n)}_p(A_1,\cdots,A_n)\left[\frac{\partial}{\partial u^{j_1}},\cdots,\frac{\partial}{\partial u^{j_k}},\frac{\partial}{\partial y_{\alpha_1}},\cdots,\frac{\partial}{\partial y_{\alpha_\ell}},\frac{\partial}{\partial\bar{y}_{\beta_1}},\cdots,\frac{\partial}{\partial\bar{y}_{\beta_{\overline{\ell}}}}\right]$$
$$\equiv\left(-\frac{1}{2\pi i}\right)^{3g-3+n}\left\langle\mathcal{B}\left[\frac{\partial}{\partial u^{j_1}}\right]\cdots\mathcal{B}\left[\frac{\partial}{\partial u^{j_k}}\right](-\partial\xi\left(y_{\alpha_1}\right))\cdots(-\partial\xi\left(y_{\alpha_\ell}\right))\right.$$

$$\left(-\overline{\partial}\overline{\xi}\left(\bar{y}_{\beta_1}\right)\right)\cdots\left(-\overline{\partial}\overline{\xi}\left(\bar{y}_{\beta_{\bar{\ell}}}\right)\right)\prod_{\alpha=l+1}^{N_R}\mathcal{X}\left(y_\alpha\right)\prod_{\beta=\bar{l}+1}^{N_L}\overline{\mathcal{X}}\left(\bar{y}_\beta\right)A_1\cdots A_n\Bigg|_{\Sigma_{g,n}}. \tag{58}$$

Of course, the insertions of $\partial\xi$ and $\overline{\partial\xi}$ each carry as much picture number as a PCO. Each insertion $(-\partial\xi\left(y_\alpha\right))$ appearing above is associated with the form $(-\partial\xi\left(y_\alpha\right))\,\mathrm{d}y_\alpha$ and is conformal invariant. Note that only the positions that have no associated tangent vector in the form have a PCO insertion.

当然，$\partial\xi$ 和 $\overline{\partial\xi}$ 的每个插入带来的图数都和一个 PCO 相同。上述每个插入 $(-\partial\xi\left(y_\alpha\right))$ 都对应形式 $(-\partial\xi\left(y_\alpha\right))\,\mathrm{d}y_\alpha$ ，并且是共形不变的。注意只有形式中未关联切向量的位置才存在 PCO 插入。

The bundle $\widehat{\mathcal{P}}^s_{g,n}$ admits subspaces $\mathcal{F}^s_{g,n}$, with $\mathcal{F}^s_{g,n} \to \mathcal{M}_{g,n}$ a degree one map, that one could hope to integrate $\Omega^{(g,n)}_{6g-6+2n}$ over to define the string amplitude. We also need to sum over spin structures—different choices of boundary conditions on GSO odd fields along homologically non-trivial cycles of the Riemann surface. It will be understood that integration over $\mathcal{F}^s_{g,n}$ includes sum over spin structures. An additional complication, however, arises because correlation functions of vertex operators on a Riemann surface suffer from spurious poles away from degeneration of the Riemann surface. These could arise, for example, when a pair of PCOs collide but, more surprisingly, can also arise on complex codimension one subspaces of the moduli space where no operators are coincident [41]. Therefore, the loci of the spurious poles span complex codimension one subspaces of $\widehat{\mathcal{P}}^s_{g,n}$. It is not clear how to choose an $\mathcal{F}^s_{g,n}$ that avoids these spurious poles. One simplification arises because the location of these poles is independent of the type of vertex operators inserted at the punctures, except for their picture numbers. Thus, a choice that avoids these poles will avoid them for all amplitudes. We shall now introduce the notion of vertical integration [42, 43] that is used to avoid the loci of all spurious poles and then comment on the nature of the resulting space $\widetilde{\mathcal{F}}^s_{g,n}$ for integration.

从 $\widehat{\mathcal{P}}^s_{g,n}$ 存在子空间 $\mathcal{F}^s_{g,n}$ ，其中 $\mathcal{F}^s_{g,n} \to \mathcal{M}_{g,n}$ 是一次映射，我们有望通过在 $\Omega^{(g,n)}_{6g-6+2n}$ 上积分来定义弦振幅。我们还需要对自旋结构求和——即 GSO 奇场在黎曼曲面同调非平凡闭链上不同的边界条件选择。后文默认对 $\mathcal{F}^s_{g,n}$ 的积分已经包含对自旋结构的求和。不过还有一个额外问题: 顶点算符在黎曼曲面上的关联函数会在黎曼曲面退化点之外出现伪极点。例如，伪极点可以出现在两个 PCO 碰撞时，但更值得注意的是，即便算符不重合，伪极点也可以出现在模空间的复余维 1 子空间上 [41]。因此伪极点的轨迹构成了 $\widehat{\mathcal{P}}^s_{g,n}$ 的复余维 1 子空间。目前尚不清楚如何选取一个能避开这些伪极点的 $\mathcal{F}^s_{g,n}$ 。一个简化之处是，这些极点的位置只和孔点处插入顶点算符的图数有关，和顶点算符的类型无关。因此只要选取的方案能避开这些极点，就可以对所有振幅都避开它们。我们现在将介绍用于避开所有伪极点轨迹的垂直积分概念 [42, 43] ，然后讨论最终得到的积分空间 $\widetilde{\mathcal{F}}^s_{g,n}$ 的性质。

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[4] The subscript n in $\widehat{\mathcal{P}}^s_{g,n}$ should really be thought of as the collection $\left(n_{-1,-1}, n_{-1,-1/2}\ , n_{-1/2,-1}, n_{-1/2,-1/2}\right)$. The locations $y_\alpha, \bar{y}_\beta$ can be chosen independently for each choice of $\left(n_{-1,-1}, n_{-1,-1/2}, n_{-1/2,-1}, n_{-1/2,-1/2}\right)$, even in cases when the (N_L, N_R) computed from (56) are equal for such choices.

[4] 下标 n 在 $\widehat{\mathcal{P}}^s_{g,n}$ 中实际上应被视为集合 $\left(n_{-1,-1}, n_{-1,-1/2}\ 、\ n_{-1/2,-1}, n_{-1/2,-1/2}\right)$ 。位置 $y_\alpha, \bar{y}_\beta$ 可以针对 $\left(n_{-1,-1}, n_{-1,-1/2}, n_{-1/2,-1}, n_{-1/2,-1/2}\right)$ 的每种选择独立选取，即便对于这些选择，由式 (56) 计算得到的 (N_L, N_R) 相等的情况也依然成立。

As a first step, one divides $\mathcal{F}^s_{g,n}$ into small cells, and then within each cell, if there are spurious poles, one modifies the PCO locations to avoid them. This can always be done. As a result of this modification, however, the positions of the PCO's will typically fail to agree at the boundaries between two cells. The boundaries we are speaking about are of dimension ($\dim \mathcal{M}_{g,n} - 1$); they are codimension one boundaries.

第一步，将 $\mathcal{F}^s_{g,n}$ 划分为多个小胞腔，若胞腔内存在伪极点，就调整 PCO 位置避开极点，这一操作总是可行的。但经过调整后，PCO 的位置通常在两个胞腔的边界处无法一致。此处所说的边界维度为 ($\dim \mathcal{M}_{g,n} - 1$)，是余维数为 1 的边界。

Consider now a boundary between two cells. As a second step, one interpolates between the choice of PCO's by using a "vertical segment"-a one-dimensional space erected over each point on the boundary between the cells. It is called vertical since along such a segment, the PCO locations change keeping the moduli of the Riemann surface and the local coordinates around the punctures fixed [42]. In other words, along such a segment, we move on a fiber of $\widehat{\mathcal{P}}^s_{g,n}$ that projects to the same point in $\widehat{\mathcal{P}}_{g,n}$.

现在考虑两个胞腔之间的边界。第二步，通过“竖直线段”在不同的 PCO 选择之间插值——竖直线段是架设在边界每个点上方的一维空间，之所以称为“竖直”，是因为沿该线段改变 PCO 位置时，黎曼面的模和穿刺周围的局部坐标都保持固定 [42]。换句话说，沿该线段，我们在 $\widehat{\mathcal{P}}^s_{g,n}$ 的一根纤维上移动，该纤维投影到 $\widehat{\mathcal{P}}_{g,n}$ 中的同一点。

The vertical segment is constructed by dividing it into several segments, such that along each segment, only one of the PCO locations change, keeping the other PCOs fixed. This makes the vertical segment one-dimensional. Thus, for example, if (y_1, y_2) are the locations of two PCOs in one cell and (y_1', y_2') are the locations of the PCOs in a neighboring cell, then the vertical segment interpolating between them could consist of the segments $(y_1, y_2) \to (y_1', y_2)$ and $(y_1', y_2) \to (y_1', y_2')$, or it could consist of the segments $(y_1, y_2) \to (y_1, y_2')$ and $(y_1, y_2') \to (y_1', y_2')$. Since $\Omega_p^{(g,n)}$ is a differential form in $\widehat{\mathcal{P}}^s_{g,n}$, we can formally integrate it along the vertical segments. In particular, since along the vertical segment only one of the PCOs change, keeping all the other PCOs and moduli fixed, it follows from (58) that the integration along the vertical direction along which the α ’ th PCO changes from y_α to y'_α produces a factor of

构造竖直线段时，先将其分为多个分段，保证每个分段仅改变一个 PCO 的位置，其余 PCO 保持固定，这样就使得整个线段是一维的。例如: 若一个胞腔内两个 PCO 的位置为 (y_1, y_2) ，相邻胞腔内 PCO 的位置为 (y_1', y_2') ，则插值的竖直线段可以由分段 $(y_1, y_2) \to (y_1', y_2)$ 和 $(y_1', y_2) \to (y_1', y_2')$ 构成，也可以由分段 $(y_1, y_2) \to (y_1, y_2')$ 和 $(y_1, y_2') \to (y_1', y_2')$ 构成。由于 $\Omega_p^{(g,n)}$ 是 $\widehat{\mathcal{P}}^s_{g,n}$ 上的微分形式，我们可以沿竖直线段对其做形式积分。特别地，因为沿竖直线段仅改变一个 PCO，其余所有 PCO 和模都保持固定，由 (58) 可知，沿竖直方向对第 α 个 PCO 从 y_α 变到 y'_α 的过程积分，会得到一个因子:

$$-\int_{y_\alpha}^{y'_\alpha} \partial\xi(y)\, dy = \xi(y_\alpha) - \xi(y'_\alpha). \tag{59}$$

This, of course, has to be further integrated along the ($\dim \mathcal{M}_{g,n} - 1$) dimensional boundary between the cells. We now see from (59) that since the result depends on the PCO locations at the two endpoints, even if there is a spurious pole along the way from y_α to y'_α , it is not seen and it does not affect the amplitude. For this argument, it is important that the correlation functions involving ξ are single valued. This follows from the results of [41].

当然，这还需要沿单元之间的 $\mathcal{M}_{g,n}-1$ 维边界进一步积分。现在我们从 (59) 可以看出，由于结果依赖于两个端点处的 PCO 位置，因此即使从 y_α 到 y'_α 的路径上存在伪极点，它也不会被观测到，也不会影响振幅。对于这个推导，包含 ξ 的关联函数是单值的这一点十分重要，这可以由文献 [41] 的结果得到。

The process does not stop here, however. When a pair of real codimension one-cell boundaries meet on a real codimension two subspace, the two vertical segments over each point on the codimension two subspace may not match, and we have to fill the gap further by erecting two dimensional vertical segments. The procedure continues to higher codimension until all the gaps are filled. A systematic procedure for doing this has been described in [43]. The end result, of course, is a space $\widetilde{\mathcal{F}}^s_{g,n}$ that is not a section in $\widehat{\mathcal{P}}^s_{g,n}$, even if the original space $\mathcal{F}^s_{g,n}$ was.

但这一过程并未就此结束。当一对实余维 1 单元边界交汇于一个实余维 2 子空间时，余维 2 子空间上每个点对应的两条竖直线段可能无法匹配，我们需要进一步通过添加二维竖直线段来填补空隙。这个过程会持续到更高余维，直到所有空隙都被填补。文献 [43] 已经描述了完成这一步骤的系统方法。当然，最终结果是，即使初始空间 $\mathcal{F}^s_{g,n}$ 是截面，得到的空间 $\widetilde{\mathcal{F}}^s_{g,n}$ 也不是 $\widehat{\mathcal{P}}^s_{g,n}$ 中的截面。

It is possible that the space $\widetilde{\mathcal{F}}^s_{g,n}$ can have "spurious" boundaries. As we move a PCO from one-point y_α to a point y'_α on a fixed Riemann surface, the curve between the two points defines the vertical segment and thus enters the description of $\widetilde{\mathcal{F}}^s_{g,n}$. On a Riemann surface, there are infinitely many homotopically inequivalent curves joining the two points. It can therefore happen that the choice of such curve, (i.e., the choice of a vertical segment) cannot be done continuously over $\widehat{\mathcal{P}}_{g,n}$. In that case, we get boundaries in $\widetilde{\mathcal{F}}^s_{g,n}$, corresponding to the unmatched vertical segments. Nevertheless, because of (59), these are spurious boundaries: as far as the amplitudes are concerned, the boundaries are effectively matched. There are no amplitude discontinuities.

空间 $\widetilde{\mathcal{F}}^s_{g,n}$ 有可能存在“伪”边界。当我们在固定黎曼曲面上将一个 PCO 从点 y_α 移动到点 y'_α 时，两点之间的曲线定义了竖直线段，并因此进入 $\widetilde{\mathcal{F}}^s_{g,n}$ 的描述。在黎曼曲面上，连接两点存在无穷多拓扑不等价的曲线。因此这类曲线的选择 (即竖直线段的选择) 可能无法在 $\widehat{\mathcal{P}}_{g,n}$ 上连续进行。这种情况下，我们会在 $\widetilde{\mathcal{F}}^s_{g,n}$ 中得到对应于未匹配竖直线段的边界。尽管如此，根据 (59)，这些都是伪边界：就振幅而言，这些边界实际上是匹配的，不会出现振幅不连续的问题。

The amplitudes in the heterotic string theory combine the features of bosonic string theory on the left-moving sector of the world-sheet and that of type II string theory on the right-moving sector of the world-sheet. This means that the additional fiber coordinates of $\widehat{\mathcal{P}}^s_{g,n}$ are only the holomorphic locations y_α of the operators changing the picture. Therefore, in (58), we drop the factors involving $\overline{\mathcal{X}}\left(\bar{y}_\alpha\right)$ and $\overline{\partial\xi}\left(\bar{y}_\alpha\right)$. Otherwise, the expressions remain the same. For a p form, with $p=k+\ell$, we have

杂化弦理论的振幅结合了世界面左行 sector 的玻色弦理论特征，和世界面右行 sector 的 II 型弦理论特征。这意味着 $\widehat{\mathcal{P}}^s_{g,n}$ 的额外纤维坐标仅为改变图景的算符的全纯位置 y_α 。因此我们在 (58) 中去掉了包含 $\overline{\mathcal{X}}\left(\bar{y}_\alpha\right)$ 和 $\overline{\partial\xi}\left(\bar{y}_\alpha\right)$ 的因子，除此之外表达式保持不变。对于满足 $p=k+\ell$ 的 p 形式，我们有

$$\Omega_p^{(g,n)}\left(A_1,\cdots,A_n\right)\left[\frac{\partial}{\partial u^{j_1}},\cdots,\frac{\partial}{\partial u^{j_k}},\frac{\partial}{\partial y_{\alpha_1}},\cdots,\frac{\partial}{\partial y_{\alpha_\ell}}\right]$$

$$= \left(-\frac{1}{2\pi i}\right)^{3g-3+n} \Big\langle \mathcal{B}\left[\frac{\partial}{\partial u^{j_1}}\right] \cdots \mathcal{B}\left[\frac{\partial}{\partial u^{j_k}}\right] \left(-\partial\xi\left(y_{\alpha_1}\right)\right) \cdots \tag{60}$$

$$\left(-\partial\xi\left(y_{\alpha_\ell}\right)\right) \prod_{\alpha=l+1}^{N_R} \mathcal{X}\left(y_\alpha\right) A_1 \cdots A_n \Big\rangle_{\Sigma_{g,n}} .$$

Amplitudes in Open-Closed String Theory

开-闭弦理论中的振幅

When open strings are present in the theory, there are some additional details to consider. First of all, we have to include Riemann surfaces with boundaries. Let us denote by $\mathcal{M}_{g,b,n_c,n_o}$ the moduli space of Riemann surfaces of genus g with b boundaries, n_c closed string punctures, and n_o open string punctures (punctures lying on boundary components). The real dimensionality d_{g,b,n_o,n_c} of $\mathcal{M}_{g,b,n_c,n_o}$ is given by

当理论中存在开弦时, 需要考虑一些额外细节。首先, 我们必须引入带边界的黎曼曲面。记 $\mathcal{M}_{g,b,n_c,n_o}$ 为亏格为 g 、带有 b 个边界、n_c 个闭弦 puncture 和 n_o 个开弦 puncture(puncture 位于边界分支上) 的黎曼曲面的模空间。$\mathcal{M}_{g,b,n_c,n_o}$ 的实维数 d_{g,b,n_o,n_c} 由下式给出

$$d_{g,b,n_c,n_o} \equiv \dim_{\mathbb{R}}\left(\mathcal{M}_{g,b,n_c,n_o}\right) = 6g - 6 + 3b + 2n_c + n_o. \tag{61}$$

Moreover, the Euler number of the type (g, b, n_c, n_o) surfaces is

此外，(g, b, n_c, n_o) 型曲面的欧拉数为

$$\chi_{g,b,n_c,n_o} = 2 - 2g - n_c - b - \frac{1}{2}n_o, \tag{62}$$

As before, since operators to be inserted need not be conformal invariant, we have to introduce the space $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ with local coordinates around all closed string punctures and all open string punctures. We discussed local coordinates around closed string punctures above. A local coordinate w at an open string puncture can be described as an analytic map from a half-disk $|w| \leq 1, \operatorname{Im}(w) \geq 0$ to a domain on the Riemann surface surrounding the puncture, with $w = 0$ mapping to the puncture and the real interval $-1 \leq w \leq 1$ mapping to a piece of the boundary of the Riemann surface that contains the puncture. Second, even for fixed g, b, n_c, n_o , the moduli space $\mathcal{M}_{g,b,n_c,n_o}$ contains disconnected components, differing from each other in the distribution of the n_o open string vertex operators among the different boundaries, and their cyclic ordering on any given boundary. Consequently, there will also be disconnected components of $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$.

和之前一样，由于待插入的算符不一定是共形不变的，我们必须对所有闭弦 puncture 和所有开弦 puncture 引入带局部坐标的空间 $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 。我们之前已经讨论过闭弦 puncture 周围的局部坐标。开弦 puncture 处的局部坐标 w 可以描述为从半圆盘 $|w| \leq 1, \operatorname{Im}(w) \geq 0$ 到黎曼曲面上包围该 puncture 的区域的解析映射，其中 $w = 0$ 映射到 puncture，实区间 $-1 \leq w \leq 1$ 映射到黎曼曲面包含该 puncture 的一段边界。其次，即使固定 g, b, n_c, n_o ，模空间 $\mathcal{M}_{g,b,n_c,n_o}$ 仍包含多个不连通分支，这些分支的区别在于 n_o 个开弦顶点算符在不同边界上的分布，以及它们在任意给定边界上的循环顺序。因此，$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 也会存在不连通分支。

The construction of forms in the presence of the open string punctures requires some new ingredients. In addition to having disks D_a around closed string punctures with gluing circles as boundaries, the relevant surfaces have half-disks d_α around open string punctures whose diameters form part of a boundary of the Riemann surface and whose semi-circular circumference gives a "gluing segment." Furthermore, in addition to having spheres S_i with three holes whose boundaries are three gluing circles, we also have hexagonal disks H_β with three boundary segments and three gluing segments arranged alternatively, annuli A_γ with one gluing circle and one boundary circle and annuli $\bar{A}_\delta$ with one gluing circle, and the other circle comprising one boundary segment and one gluing segment. These components are joined across gluing circles C_s and gluing segments L_m to form the Riemann surface. We shall denote the numbers of various constituents of a Riemann surface by #D, #S, #d, $\#H, \#A, \#\bar{A}, \#C$, and $\#L$. Of these, $\#D$ is equal to the number n_c of external closed string punctures, and $\#d$ is equal to the number n_o of external open string punctures. An example of such a construction has been shown in Fig. 3.

在存在开弦 puncture 的情况下构造形式需要一些新要素。除了闭弦 puncture 周围以粘合圆为边界的圆盘 D_a 外，相关曲面还在开弦 puncture 周围带有半圆盘 d_α ， 半圆盘的直径构成黎曼曲面边界的一部分，半圆盘的半圆周给出“粘合段”。此外，除了带有三个孔、边界为三个粘合圆的球面 S_i ，我们还会遇到交替排列三个边界段和三个粘合段的六角形圆盘 H_β 、带有一个粘合圆和一个边界圆的环面 A_γ ，以及带有一个粘合圆、另一个圆包含一段边界和一个粘合段的环面 $\bar{A}_\delta$ 。这些组分通过粘合圆 C_s 和粘合段 L_m 连接起来构成黎曼曲面。我们用 #D、#S、#d、$\#H, \#A, \#\bar{A}, \#C$ 和 $\#L$ 表示黎曼曲面各组分的数量。其中，$\#D$ 等于外部闭弦 puncture 的数量 n_c ，$\#d$ 等于外部开弦 puncture 的数量 n_o 。图 3 给出了这种构造的一个例子。

There are some constraints on the various numbers. From the Euler number of the surfaces, we find that: [5]

各类组分的数量存在一些约束。由曲面的欧拉数，我们得到: [5]

$$\#S + \frac{1}{2}\#H + \frac{1}{2}\#\bar{A} = -\chi_{g,b,n_c,n_o}. \tag{63}$$

[5] For the purpose of computing the Euler number, we can replace a gluing circle by a closed string puncture and a gluing segment by an open string puncture. For example, a three holed sphere S can be regarded as a sphere with three closed string punctures, a hexagon H can be regarded as a disk with three open string punctures, an annulus A can be regarded as a disk with one closed string puncture, an annulus A can be regarded as a disk with one closed string puncture and one open string puncture, a disk D can be regarded as a sphere with two closed string punctures, and a half-disk d can be regarded as a disk with two open string punctures. This gives the Euler numbers of S to be $-1, H$ to be $-1/2, \bar{A}$ to be $-1/2$, and A, D , and d to be zero.

[5] 为了计算欧拉示性数，我们可以将粘合圆替换为闭弦穿孔，将粘合线段替换为开弦穿孔。例如，三孔球面 S 可视为带有三个闭弦穿孔的球面，六边形 H 可视为带有三个开弦穿孔的圆盘，环面 A 可视为带有一个闭弦穿孔的圆盘，环面 A 可视为带有一个闭弦穿孔和一个开弦穿孔的圆盘，圆盘 D 可视为带有两个闭弦穿孔的球面，半圆盘 d 可视为带有两个开弦穿孔的圆盘。由此可得 S 的欧拉示性数为 $-1, H$ ， $-1, H$ 的为 $-1/2, \bar{A}$ ， $-1/2, \bar{A}$ 的为 $-1/2$ ， A, D 的为， d 的为零。

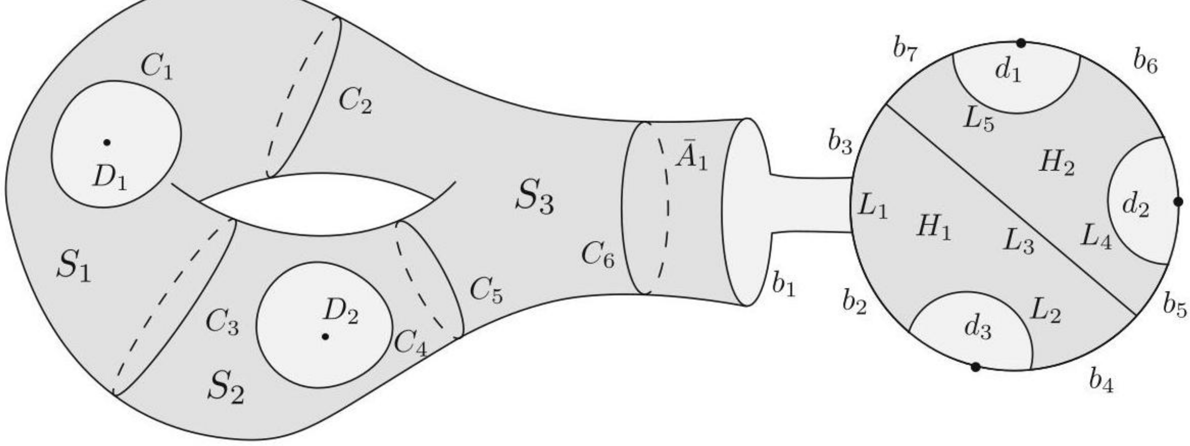

Fig. 3 A genus one surface with one boundary, two closed string punctures ($n_c = 2$) and three open string punctures ($n_o = 3$) . It is built with two disks D_1, D_2 , three half-disks d_1, d_2, d_3 , three spheres S_1, S_2, S_3 , one annulus $\bar{A}_1$, and two hexagonal disks H_1, H_2 . These pieces are joined across six gluing circles $C_1, \cdots, C_6$ and five gluing segments $L_1, \cdots L_5$. The single boundary is composed of b_i ’ s as well as segments of d_i ’ s containing the open string punctures (This figure is from [22])

图 3 亏格为 1、带有 1 个边界、2 个闭弦穿孔 ($n_c = 2$) 和 3 个开弦穿孔 ($n_o = 3$) 的曲面。它由 2 个圆盘 D_1, D_2 、3 个半圆盘 d_1, d_2, d_3 、3 个球面 S_1, S_2, S_3 、1 个环面 $\bar{A}_1$ 和 2 个六边形圆盘 H_1, H_2 构成。这些片通过 6 个粘合圆 $C_1, \cdots, C_6$ 和 5 个粘合线段 $L_1, \cdots L_5$ 连接。唯一的边界由多个 b_i 以及包含开弦穿孔的 d_i 段组成 (本图引自 [22])

On the left-hand side, each term is (minus) the contribution to the Euler number from each component of the surface. Furthermore, since each gluing circle C_i is shared by two components (or two gluing circles of the same components), we have

左侧每一项是 (负的) 曲面各分支对欧拉示性数的贡献。此外，由于每个粘合圆 C_i 都为两个分支 (或同一分支的两个粘合圆) 共有，我们得到

$$2\#C = 3\#S + n_c + \#A + \#\bar{A}. \tag{64}$$

A similar result holds for gluing segments:

对于粘合线段也有类似结论:

$$2\#L = 3\#H + n_o + \#\bar{A}. \tag{65}$$

As a simple check, for surfaces without boundaries and thus no open string punctures, one quickly sees that $\#S = 2g - 2 + n_c$ and that $\#C = 3g - 3 + 2n_c$, familiar results from closed string field theory. It should be noted that for open-closed theory, a given surface may be built with different number of ingredient regions. For example, an annulus with one open string puncture can be built with either $\{A_1, \bar{A}_1, d_1\}$ or with $\{H_1, d_1\}$.

作为一个简单验证，对于没有边界因此也没有开弦穿孔的曲面，我们可以很快得到 $\#S=2g-2+n_c$ 和 $\#C=3g-3+2n_c$ ，这是闭弦场论中已知的结论。需要注意的是，对于开-闭弦理论，一个给定曲面可以由不同数量的基础区域构造。例如，带一个开弦穿孔的环面既可以用 $\{A_1,\bar{A}_1,d_1\}$ 构造，也可以用 $\{H_1,d_1\}$ 构造。

In this case, the antighost insertion (43) will have additional contributions in the form of integrals over gluing segments L_m separating the different coordinate patches:

这种情况下，反鬼插入 (43) 会产生额外贡献，形式为分隔不同坐标补丁的粘合线段 L_m 上的积分:

$$\mathcal{B}\left[\frac{\partial}{\partial u^i}\right]\equiv\sum_s\left[\oint_{C_s}\frac{\partial F_s}{\partial u^i}d\sigma_s b\left(\sigma_s\right)+\oint_{C_s}\frac{\partial\bar{F}_s}{\partial u^i}d\bar{\sigma}_s\bar{b}\left(\bar{\sigma}_s\right)\right]$$

$$+\sum_m\left[\int_{L_m}\frac{\partial G_m}{\partial u^i}d\sigma_m b\left(\sigma_m\right)+\int_{L_m}\frac{\partial\bar{G}_m}{\partial u^i}d\bar{\sigma}_m\bar{b}\left(\bar{\sigma}_m\right)\right],\tag{66}$$

where we define transition functions

其中我们定义转移函数

$$\sigma_m=G_m\left(\tau_m,u^i\right),\tag{67}$$

that express the complex coordinate σ_m to the left of L_m in terms of the complex coordinate τ_m to the right of L_m , given the orientation of L_m that goes into the definition of the integral in (66). This choice defines signs in explicit calculations. With this understanding, for a given set of n_c closed string states and n_o open string states, we can introduce a p form $\Omega_p^{(g,b,n_c,n_o)}$ on $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ as in (44), for the case of bosonic open-closed amplitudes, and as in (58), for the case of open-closed superstring amplitudes. However, some care is needed to specify the normalization of these forms since they are somewhat more involved than in the case of purely closed string amplitudes. For reference, we shall discuss the bosonic string p -forms with n_c closed string vertex operators and n_o open string vertex operators. We first define the canonical forms $\widehat{\Omega}_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots A_{n_o}^o\right)$ denoted with a hat. The contraction of $\widehat{\Omega}_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots A_{n_o}^o\right)$ with p tangent vectors of $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ is taken to be:

它们根据 (66) 中积分定义所采用的 L_m 方向, 将 L_m 左侧的复坐标 σ_m 表示为 L_m 右侧复坐标 τ_m 的函数。该选择会确定显式计算中的符号。基于这一前提, 对于给定的一组 n_c 个闭弦态和 n_o 个开弦态, 我们可以如玻色开-闭弦振幅情形的 (44)、以及开-闭超弦振幅情形的 (58) 那样, 在 $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 上引入一个 p 形式 $\Omega_p^{(g,b,n_c,n_o)}$ 。不过，由于这些形式比纯闭弦振幅的情况更复杂, 我们需要仔细确定它们的归一化。作为参考, 我们将讨论带有 n_c 个闭弦顶点算子和 n_o 个开弦顶点算子的玻色弦 p 形式。我们首先定义带 hat 标记的正则形式 $\widehat{\Omega}_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots A_{n_o}^o\right)$ 。$\widehat{\Omega}_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots A_{n_o}^o\right)$ 与 $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 的 p 个切向量的缩并定义为:

$$\widehat{\Omega}_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right)\left[\frac{\partial}{\partial u^{j_1}},\cdots,\frac{\partial}{\partial u^{j_p}}\right]$$

$$\sim\left\langle\mathcal{B}\left[\frac{\partial}{\partial u^{j_1}}\right]\cdots\mathcal{B}\left[\frac{\partial}{\partial u^{j_p}}\right]A_1^c\cdots A_{n_c}^c;A_1^o\cdots A_{n_o}^o\right\rangle_{\Sigma_{g,b,n_c,n_o}},\tag{68}$$

where $\sim$ denotes equality up to sign. Here, the surface $\sum_{g,b,n_c,n_o}$ has b boundary components, with the n_o punctures distributed among them as $(n_o^1,\ldots,n_o^b)$ so that $\sum_{i=1}^{b} n_o^i = n_o$. The $\widehat{\Omega}$ ' s defined above suffer from sign ambiguities, there being no canonical choice for the sign of the integration measure $du^{j_1}\wedge\cdots\wedge du^{j_p}$. We shall fix this problem in section "Signs of Forms in $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ " by choosing a specific convention for the sign of the integration measure and specific arrangement of the operators $\mathcal{B},A_i^c$ and A_j^o inside the correlator that will also depend on the Grassmann parities of the operators A_i^c and A_i^o . Normalized forms $\Omega_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right)$ are then given by canonical forms $\widehat{\Omega}$ multiplied by a normalization factor:

其中 $\sim$ 表示相差符号意义下的相等。此处曲面 $\sum_{g,b,n_c,n_o}$ 有 b 个边界分支，n_o 个 puncture 按 $(n_o^1,\ldots,n_o^b)$ 分布在各边界上，满足 $\sum_{i=1}^{b} n_o^i = n_o$ 。上述定义的 $\widehat{\Omega}$ 存在符号歧义，因为积分测度 $du^{j_1}\wedge\cdots\wedge du^{j_p}$ 没有标准的符号选择。我们将在“$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 中形式的符号”一节通过选择积分测度符号的特定约定，以及关联子中算子 $\mathcal{B},A_i^c$ 和 A_j^o 的特定排列来解决这个问题，排列方式还会依赖算子 A_i^c 和 A_i^o 的格拉斯曼奇偶性。归一化形式 $\Omega_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right)$ 即为正则形式 $\widehat{\Omega}$ 乘以下列归一化因子:

$$\Omega_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right)$$

$$\equiv N_{g,b,n_c,n_o}\widehat{\Omega}_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right). \tag{69}$$

Normalization constants N_{g,b,n_c,n_o} are the analogs of the $\left(-\frac{1}{2\pi i}\right)^{3g-3+n}$ factors in (44). With the specific prescription for the signs of $\widehat{\Omega}$ discussed in section "Signs of Forms in $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ ", the result for N_{g,b,n_c,n_o} is given by

归一化常数 N_{g,b,n_c,n_o} 是 (44) 式中 $\left(-\frac{1}{2\pi i}\right)^{3g-3+n}$ 因子的类似物。结合“$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 中形式的符号”一节讨论的 $\widehat{\Omega}$ 符号的特定规定，可得到 N_{g,b,n_c,n_o} 的结果为

$$N_{g,b,n_c,n_o} = \eta_c^{3g-3+n_c+\frac{3}{2}b+\frac{3}{4}n_o},\ \eta_c \equiv -\frac{1}{2\pi i} = \frac{i}{2\pi}. \tag{70}$$

The form (69) for $p = d_{g,b,n_c,n_o}$ has to be integrated over a subspace $\mathcal{F}_{g,b,n_c,n_o}$ for which the projection to the moduli space $\mathcal{M}_{g,b,n_c,n_o}$ is a degree one map. The space $\mathcal{F}_{g,b,n_c,n_o}$ consists of disconnected components, one for each disconnected component of $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$. Only after summing over the contributions from these disconnected components with appropriate relative signs can we recover an amplitude that is symmetric under the exchange of external open string states. The amplitude $\mathcal{A}_{g,b}$ takes the form:

$p = d_{g,b,n_c,n_o}$ 的 (69) 式形式必须在子空间 $\mathcal{F}_{g,b,n_c,n_o}$ 上积分，该子空间到模空间 $\mathcal{M}_{g,b,n_c,n_o}$ 的投影是一次映射。空间 $\mathcal{F}_{g,b,n_c,n_o}$ 由不连通分支组成，每个分支对应 $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 的一个不连通分支。只有对这些不连通分支的贡献按适当相对符号求和后，我们才能得到在外开弦态交换下对称的振幅。振幅 $\mathcal{A}_{g,b}$ 形式如下:

$$\mathcal{A}_{g,b}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right)$$

$$= (g_s)^{-\chi_{g,b,n_c,n_o}} \int_{\mathcal{F}_{g,b,n_c,n_o}} \Omega^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}} \left(A^c_1, \cdots, A^c_{n_c}; A^o_1, \cdots, A^o_{n_o}\right), \tag{71}$$

where the integral over $\mathcal{F}_{g,b,n_c,n_o}$ is understood to include the sum over disconnected components of $\mathcal{F}_{g,b,n_c,n_o}$.

其中对 $\mathcal{F}_{g,b,n_c,n_o}$ 的积分默认包含对 $\mathcal{F}_{g,b,n_c,n_o}$ 不连通分支的求和。

An exception to (68) arises for the closed string one-point function on the disk where we have a conformal Killing vector. Here we take, for Grassmann even A^c , and without sign ambiguity

(68) 式存在一个例外: 圆盘上的闭弦单点函数，该情形下存在共形 Killing 矢量。对于格拉斯曼偶 A^c ，不存在符号歧义，我们取

$$\widehat{\Omega}^{(0,1,1,0)}_0 (A^c) = \eta_c \langle c_0^- A^c \rangle, \tag{72}$$

where $c_0^- A^c$ is inserted in a local coordinate system w in which the disk boundary is at $|w| = e^{\Lambda}$ for some positive constant Λ . Consequently the form of (69) gives:

其中 $c_0^- A^c$ 被插入局部坐标系 w ，该坐标系中圆盘边界位于 $|w| = e^{\Lambda}$ ，Λ 为某正常数。因此 (69) 式的形式给出:

$$\Omega^{(0,1,1,0)}_0 (A^c) = \eta_c N_{0,1,1,0} \langle c_0^- A^c \rangle. \tag{73}$$

We could have included the η_c factor in $N_{0,1,1,0}$, but in the form given above,(70) will also hold for $N_{0,1,1,0}$.

我们本可以把 η_c 因子包含在 $N_{0,1,1,0}$ 中，但就上面给出的形式而言，(70) 式对 $N_{0,1,1,0}$ 依然成立。

From the forms defined above, we can derive a result similar to (51) for open string amplitudes. Let us define

从上述定义的形式出发，我们可以推导出开弦振幅一个类似 (51) 式的结果。我们定义

$$c_o \equiv N_{g,b,n_c,n_o+1} / N_{g,b,n_c,n_o}. \tag{74}$$

We see from (70) that c_o does not depend on g, b, n_c, n_o . Let us now study the effect of inserting an on-shell open string state of the form cV_o to some given amplitude. For this, let w be the local coordinate at the puncture where the open string is inserted, with the puncture situated at $w = 0$, and

从 (70) 式可以看出 c_o 不依赖于 g, b, n_c, n_o 。现在我们来研究在给定振幅中插入形式为 cV_o 的在壳开弦态的效应。为此，设 w 是开弦插入点孔处的局部坐标，孔位于 $w = 0$ ，且

$$z = G(w, u), \tag{75}$$

be the coordinate system in a patch of the surface that encloses the open set covered by the w coordinate. We shall assume that $G(w,u)$ is analytic in the half disk covered by the w coordinate. Then the location y of the puncture in the z coordinate system is $y = G(0,u)$. Up to additional insertions (denoted by dots) that we do not focus on, the effect of inserting the vertex operator cV_o is represented by the one-form

是曲面一个邻域上的坐标系，该邻域覆盖了 w 坐标覆盖的开集。我们假设 $G(w,u)$ 在 w 坐标覆盖的半圆盘内解析。那么孔在 z 坐标系中的位置 y 为 $y = G(0,u)$ 。暂不关注额外插入项 (用点号表示)，插入顶点算符 cV_o 的效应可以用下述一元形式表示:

$$\Omega_1(cV_o) = c_o du \left\langle \cdots \mathcal{B}\left[\frac{\partial}{\partial u}\right] cV_o(w=0) \right\rangle. \tag{76}$$

We now use the doubling trick to express the sum of the holomorphic and anti-holomorphic integrals along open contours in $\mathcal{B}$ into integration over a closed contour surrounding the puncture at $w = 0$ or equivalently at $z = y$. Since cV_o is a dimension zero primary, we have $cV_o(w=0) = cV_o(z=y)$, and therefore we have

我们现在使用加倍技巧，将 $\mathcal{B}$ 中沿开围道的全纯积分和反全纯积分之和表达为绕位于 $w = 0$ (等价于位于 $z = y$) 的孔的闭围道积分。由于 cV_o 是零维主元，故有 $cV_o(w=0) = cV_o(z=y)$ ，因此我们得到

$$\Omega_1(cV_o) = c_o du \left\langle \cdots \oint b(z)\, dz \frac{\partial G(w,u)}{\partial u} cV_o(z=y) \right\rangle. \tag{77}$$

Since we have prescribed that the w coordinate system must be to the right of the contour, the contour is oriented clockwise around the position of the puncture. The integral then gives

由于我们规定 w 坐标系必须在围道的右侧，因此围道沿顺时针方向环绕孔的位置。积分结果为

$$\Omega_1(cV_o) = -c_o du \frac{\partial y}{\partial u} \langle \cdots V_o(z=y) \rangle = -c_o dy \langle \cdots V_o(y) \rangle. \tag{78}$$

Here $V_o(y)$ is to be understood as the vertex operator at $z = y$ using the local coordinate $(z-y)$ vanishing at that insertion point.

此处 $V_o(y)$ 应理解为 $z = y$ 处利用在插入点处为零的局部坐标 $(z-y)$ 定义的顶点算符。

In arriving at the last expression in (78), we had the operator $\mathcal{B}$ in (76) placed immediately to the left of the vertex operator cV_o . In general, however, we have to move the $\mathcal{B}$ through various other vertex operators and other $\mathcal{B}$'s, leading to additional factors of -1 . Therefore (78) is valid up to a sign. This is not a serious limitation since in any case, the sign of N_{g,b,n_c,n_o} , and hence of c_o , is ambiguous until we fix the sign conventions for $\widehat{\Omega}$, and we shall fix this convention in section "Signs of Forms in $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ ". This issue was absent in the case of closed string punctures since there the combination that appeared is $\mathcal{B}[\partial/\partial u]\, \mathcal{B}[\partial/\partial \bar{u}]$, and no extra sign appears as we move this combined operator through other operators.

在得到 (78) 的最终表达式时，我们将 (76) 中的算符 $\mathcal{B}$ 直接放在了顶点算符 cV_o 的左侧。但一般而言，我们需要将 $\mathcal{B}$ 移过其他各类顶点算符与其他 $\mathcal{B}$ ，这会产生额外的-1 因子，因此 (78) 仅在符号范围内成立。这并不是严重的限制: 因为无论如何，在我们固定 $\widehat{\Omega}$ 的符号规范之前，N_{g,b,n_c,n_o} 乃至 c_o 的符号都是不确定的，我们会在「$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 中形式的符号」一节固定这一规范。闭弦穿孔的情况不存在这个问题，因为该处出现的组合是 $\mathcal{B}\left[\partial/\partial u\right]\mathcal{B}\left[\partial/\partial\bar{u}\right]$ ，移动这个组合算符经过其他算符时不会产生额外符号。

Finally, note that a general background in string theory may contain multiple D-branes of the same type or different type and the spectrum and interaction of open strings will depend on that data. In our description of open-closed string amplitudes, we have implicitly assumed (and will continue to assume) that there is a single D-brane on which the open strings live. This, however, can be easily generalized to the case of multiple D-branes. For example, if we have multiple D-branes of the same type, then the external open strings will carry Chan-Paton factors, and we have to take the trace of the product of Chan-Paton factors separately for every boundary of the world-sheet. When there are different types of D-branes, the analysis is a bit more complicated but follows the same route. For example, the constant K appearing in (12) may now have different values for different open string sectors. We can take this effect into account by regarding K as a matrix instead of a number, and in computing the correlator, we have to have a factor of K inserted into the trace for each boundary separately.

最后需要注意，弦论中的一般背景可以包含多个同类型或不同类型的 D 膜，开弦的谱与相互作用都会依赖这些数据。在我们对开-闭弦振幅的描述中，我们已经隐含假设 (并且会继续假设) 开弦仅存在于单个 D 膜上，但这很容易推广到多个 D 膜的情况。例如，如果存在多个同类型 D 膜，那么外部开弦会携带陈-帕顿因子，我们需要对世界面的每个边界分别对陈-帕顿因子的乘积求迹。当存在不同类型的 D 膜时，分析稍复杂但遵循相同的思路: 例如，(12) 中出现的常数 K 现在对不同的开弦扇区可以取不同的值。我们可以通过将 K 视为矩阵而非数来纳入这一效应，在计算关联函数时，我们需要对每个边界分别在迹中插入一个 K 因子。

Turning now briefly to open-closed superstring field theory, the relevant forms are a straightforward generalization of the expressions written before. We have

现在简要讨论开-闭超弦场论，相关形式是前文表达式的直接推广，我们有:

$$
\begin{aligned}
&\Omega_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right)\\
&\times\left[\frac{\partial}{\partial u^{j_1}},\cdots,\frac{\partial}{\partial u^{j_k}},\frac{\partial}{\partial y_{\alpha_1}},\cdots,\frac{\partial}{\partial y_{\alpha_\ell}},\frac{\partial}{\partial\bar{y}_{\beta_1}},\cdots,\frac{\partial}{\partial\bar{y}_{\beta_\ell}}\right]\\
&\sim N_{g,b,n_c,n_o}\Big\langle\mathcal{B}\left[\frac{\partial}{\partial u^{j_1}}\right]\cdots\mathcal{B}\left[\frac{\partial}{\partial u^{j_k}}\right]\left(-\partial\xi\left(y_{\alpha_1}\right)\right)\cdots\left(-\partial\xi\left(y_{\alpha_\ell}\right)\right)\left(-\bar{\partial}\bar{\xi}\left(\bar{y}_{\beta_1}\right)\right)\\
&\cdots\left(-\bar{\partial}\bar{\xi}\left(\bar{y}_{\beta_{\bar{\ell}}}\right)\right)\prod_{\alpha=l+1}^{N_R}\mathcal{X}\left(y_\alpha\right)\prod_{\beta=\bar{l}+1}^{N_L}\overline{\mathcal{X}}\left(\bar{y}_\beta\right)A_1^c\cdots A_{n_c}^c A_1^o\cdots A_{n_o}^o\Big\rangle_{\Sigma_{g,b,n_c,n_o}},(79 \qquad (,79)
\end{aligned}
$$

where the normalization constants N_{g,b,n_c,n_o} turn out to be the same as those for bosonic string theory and $\sim$ denotes equivalence up to sign. One point that we need to keep in mind is that once we have Riemann surfaces with boundaries, the picture numbers in the holomorphic and anti-holomorphic sectors are not separately conserved, only the total picture number is conserved. The total picture number must add to $2(2g-2+b)$, consistent with total picture -2 for a disk and -4 for a sphere. The total number N of PCO insertions, as seen in the above formula, is $N=N_R+N_L$. In the notation of (56), N is fixed by

其中归一化常数 N_{g,b,n_c,n_o} 与玻色弦理论的归一化常数相同，$\sim$ 表示相差符号意义下的等价。我们需要记住一点: 一旦我们考虑带边界的黎曼曲面，全纯和反全纯扇区的图数不再分别守恒，只有总图数守恒。总图数之和必须等于 $2(2g-2+b)$，这与圆盘总图数为-2、球面总图数为-4 一致。从上式可以看出，PCO 插入的总个数 N 为 $N=N_R+N_L$ 。在 (56) 的记号下，N 由下式固定:

$$2(2g-2+b)=N-2n_{-1,-1}-\frac{3}{2}\left(n_{-1,-1/2}+n_{-1/2,-1}\right)$$

$$-n_{-1/2,-1/2},-n_{-1}-\frac{1}{2}n_{-1/2}. \tag{80}$$

New here on the right-hand side are n_{-1} and $n_{-1/2}$ that denote, respectively, the number of NS and R (open string) vertex operators. Therefore, for a given set of punctures on a Riemann surface, we have a choice of how to distribute the N PCOs into $N_R\geq 0$ holomorphic and $N_L\geq 0$ anti-holomorphic PCOs. This information also constitutes data along the fiber of $\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$. For the closed string one-point function in open-closed superstrings, an expression similar to (73) holds. The factor $\left(-\frac{1}{2\pi i}\right)$ is also there, but the correlator includes picture changing operators for RR states in $\widetilde{\mathcal{H}}_c$ (see (158)).

右侧此处新增了 n_{-1} 和 $n_{-1/2}$，它们分别表示 NS 型和 R 型 (开弦) 顶点算子的数量。因此，对于黎曼曲面上给定的一组 puncture，我们可以选择将 N 个 PCO 分配为 $N_R\geq 0$ 个全纯 PCO 和 $N_L\geq 0$ 个反全纯 PCO。该信息也属于 $\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$ 纤维上的数据。对于开-闭超弦中的闭弦单点函数，存在与式 (73) 类似的表达式。这里也包含因子 $\left(-\frac{1}{2\pi i}\right)$，但关联函数中包含了 $\widetilde{\mathcal{H}}_c$ 里 RR 态的图变算子 (参见式 (158))。

Signs of Forms in $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$

$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 中形式的符号

In this subsection, we shall discuss a specific choice of sign convention for the $\widehat{\Omega}_p^{(g,b,n_c,n_o)}$, s that is compatible with the normalization constants N_{g,b,n_c,n_o} appearing in (69). We shall consider the case $p=d_{g,b,n_c,n_o}$ and take the arguments A_i^c and A_j^o of $\widehat{\Omega}$ to be Grassmann even and Grassmann odd, respectively, since this is what enters the construction of the string field theory action. To avoid cluttering, we shall drop the subscript d_{g,b,n_c,n_o} from $\widehat{\Omega}^{(g,b,n_c,n_o)}$. We shall briefly comment on the other cases later.

在本小节中，我们将讨论 $\widehat{\Omega}_p^{(g,b,n_c,n_o)}$ 的一种特定符号规范选择，该选择与出现在式 (69) 中的归一化常数 N_{g,b,n_c,n_o} 相容。由于这是构造弦场论作用量所需要的情形，我们将考虑 $p=d_{g,b,n_c,n_o}$ 的情况，并将 $\widehat{\Omega}$ 的自变量 A_i^c 和 A_j^o 分别取为格拉斯曼偶和格拉斯曼奇。为避免符号混乱，我们将省略 $\widehat{\Omega}^{(g,b,n_c,n_o)}$ 的下标 d_{g,b,n_c,n_o} 。我们稍后会对其他情况作简要说明。

We begin by discussing the origin of the sign ambiguities. Unlike in the case of a closed string amplitude that contains the product $\mathcal{B}\left[\partial/\partial u\right]\mathcal{B}\left[p/\partial\bar{u}\right]$, which is Grassmann even, the open string amplitude contains

$\mathcal{B}[\partial/\partial u]$, which is Grassmann odd. The vertex operators of open string states are also Grassmann odd. Therefore, we can pick up additional minus signs when we move these operators through other Grassmann odd operators, and the sign will depend on the relative arrangement of the $\mathcal{B}$ ’ s and the vertex operators inside the correlator. We have already seen an example of this in the manipulations leading to (78). A related issue is that unlike closed string moduli $m = m_R + im_I$ that are complex and hence define a natural orientation via $dm \wedge d\bar{m} = -2idm_R \wedge dm_I$ with the integral of $dm_R \wedge dm_I$ taken to be positive, for open strings, the moduli are real, and there is no natural choice of what constitutes positive integration measure. So the expression for $\widehat{\Omega}_p^{(g,b,n_c,n_o)}$ makes sense only after we have specified the positions of the $\mathcal{B}$ ’ s inside the correlator and the orientation of the moduli space measure. For this reason, we shall now specify these rules. There is nothing sacred about these rules-they give one definition of $\widehat{\Omega}^{(g,b,n_c,n_o)}$ appearing in (68) and the orientation of the moduli space integral, without which the sign of N_{g,b,n_c,n_o} will not be meaningful. For more details, we refer the reader to [22]. We shall describe these rules in the context of bosonic string theory, but generalization of these rules to superstring theory, that produce the same values of N_{g,b,n_c,n_o} as those given in (84), is straightforward.

我们首先讨论符号歧义的来源。包含乘积 $\mathcal{B}[\partial/\partial u]\,\mathcal{B}[p/\partial\bar{u}]$ 的闭弦振幅是格拉斯曼偶, 与之不同, 开弦振幅包含的 $\mathcal{B}[\partial/\partial u]$ 是格拉斯曼奇。开弦态的顶点算符也是格拉斯曼奇。因此, 当我们将这些算符移动穿过其他格拉斯曼奇算符时，会产生额外的负号，符号取决于关联函数中 $\mathcal{B}$ 与顶点算符的相对排列。我们在推导向 (78) 的操作中已经见过这样的例子。另一个相关问题是: 闭弦模 $m = m_R + im_I$ 是复的，因此可以通过 $dm \wedge d\bar{m} = -2idm_R \wedge dm_I$ 定义自然定向，且 $dm_R \wedge dm_I$ 的积分取为正，而开弦的模是实的，不存在自然选择来定义何为正积分测度。因此 $\widehat{\Omega}_p^{(g,b,n_c,n_o)}$ 的表达式只有在我们指定了关联函数中 $\mathcal{B}$ 的位置以及模空间测度的定向之后才有意义。因此，我们现在来明确这些规则。这些规则并非绝对不可变——它们给出了式 (68) 中 $\widehat{\Omega}^{(g,b,n_c,n_o)}$ 的一种定义以及模空间积分的定向，若没有这些，N_{g,b,n_c,n_o} 的符号将没有意义。更多细节读者可以参考文献 [22]。我们将在玻色弦理论的框架下描述这些规则，但将这些规则推广到超弦理论是直接的，且推广后得到的 N_{g,b,n_c,n_o} 与式 (84) 给出的结果一致。

1. First we shall describe the procedure for adding a boundary to a given amplitude. We take the expression for the form $\widehat{\Omega}^{(g,b+1,n_c,n_o)}$ to be the one that is obtained by starting with the form $\widehat{\Omega}^{(g,b,n_c+1,n_o)}$ with one more closed string puncture and, inserting into the extra puncture the one-form state $|\mathbf{b}\rangle$ given by

1. 首先我们描述给给定振幅添加边界的步骤。我们取形式 $\widehat{\Omega}^{(g,b+1,n_c,n_o)}$ 的表达式，它可以通过如下方式得到: 从多一个闭弦孔的形式 $\widehat{\Omega}^{(g,b,n_c+1,n_o)}$ 出发，在额外的孔中插入由下式给出的一元态 $|\mathbf{b}\rangle$:

$$|\mathbf{b}\rangle = -\frac{dq_r}{q_r}\left(e^{-\Lambda}q_r\right)^{L_0+\bar{L}_0} b_0^+|B\rangle,\ q_r \in [0,1]\,, \tag{81}$$

where $|B\rangle$ is the boundary state defined as follows. For any closed string state, $|\chi\rangle, \langle B|\,\chi\rangle$ gives the one-point function of χ on the unit disk, with χ inserted at $w = 0$ using the local coordinate system w in which the disk takes the form $|w| \le 1$. Since the one-point function of χ on the disk is non-vanishing only when χ has ghost number three and $\langle B \mid \chi\rangle$ is non-vanishing only when the ghost numbers of B and χ add up to six, we conclude that $|B\rangle$ has ghost number three.

其中 $|B\rangle$ 是如下定义的边界态。对任意闭弦态，$|\chi\rangle, \langle B|\,\chi\rangle$ 给出单位圆盘上 χ 的单点函数，χ 通过局部坐标系 w 插入在 $w = 0$ 处，在该坐标系中圆盘形如 $|w| \le 1$ 。由于仅当 χ 的鬼数为 3 时，圆盘上 χ 的单点函数非零，且仅当 B 和 χ 的鬼数之和为 6 时 $\langle B \mid \chi\rangle$ 非零，因此我们可得 $|B\rangle$ 的鬼数为 3。

Hence, $b_0^+|B\rangle$ is a Grassmann even state, and we can insert it anywhere inside the correlation function. One can also show that the state $|B\rangle$ is annihilated by b_0^- and L_0^- . The orientation of q_r integration is along the direction of increasing q_r , i.e., in the integral, the upper limit should be larger than the lower limit. Λ is an arbitrary positive constant the same constant that appears in the choice of the local coordinate on the disk in (72). With this understanding, the above relation fully fixes the form $\widehat{\Omega}^{(g,b+1,n_c,n_o)}$ once $\widehat{\Omega}^{(g,b,n_c+1,n_o)}$ is given. The parameter q_r , together with the location of the (n_c+1) -th puncture, comprises the three extra moduli that appear when we add a hole to the Riemann surface. This prescription can be used iteratively to add arbitrary number of boundaries starting with a Riemann surface with closed string punctures but no boundaries.

因此 $b_0^+|B\rangle$ 是格拉斯曼偶态，可插入关联函数的任意位置。还可证明态 $|B\rangle$ 被 b_0^- 和 L_0^- 零化。q_r 积分的取向沿 q_r 递增方向，即积分中上限应大于下限。Λ 是任意正常数，与 (72) 式中圆盘局部坐标选取时出现的常数一致。据此，给定 $\widehat{\Omega}^{(g,b,n_c+1,n_o)}$ 后上述关系就完全确定了形式 $\widehat{\Omega}^{(g,b+1,n_c,n_o)}$ 。参数 q_r 与第 (n_c+1) 个孔的位置共同构成三个额外模，这三个模是我们在黎曼曲面上添加一个孔时会出现的。这个方案可以迭代使用，从仅有闭弦孔、无边界的黎曼曲面出发，添加任意数量的边界。

2. Next we shall describe the $\mathcal{B}$'s associated with the moduli describing location of open string punctures. For this, suppose that we have some amplitude involving external closed and open strings and we want to insert one more external open string state. We shall first describe this for on-shell open strings with vertex operator cV_o since this is sufficient for specifying the sign. In this case, we insert into the original correlator the operator

2. 接下来我们将描述与描述开弦孔位置的模相关的 $\mathcal{B}$ 。为此，假设我们有一个包含外闭弦和外开弦的振幅，现在想要再插入一个外开弦态。我们先介绍顶点算子为 cV_o 的在壳开弦的情况，因为这足以确定符号。在这种情况下，我们向原关联函数插入算子

$$\mathrm{d}y\, V_o(y)\ , \tag{82}$$

where y is the location of the operator on a boundary and the integration along y is carried out along the boundary keeping the surface to the left. Here, $V_o(y)$ has to be interpreted in the sense described below (78). It follows from the analysis described there that the prescription (82) is equivalent to inserting $(-\mathcal{B}[\partial/\partial u])$ immediately to the left of the unintegrated vertex operator $cV_o(u)$ (or, equivalently, inserting $\mathcal{B}[\partial/\partial u]$ immediately to the right of the vertex operator). In this form, the prescription holds for any external open string states, not necessarily on-shell. Since the inserted operator is Grassmann even, there is no sign ambiguity arising from the location of the operator inside the correlation function.

其中 y 是算子在边界上的位置，沿 y 的积分沿边界进行，保持曲面在左侧。此处 $V_o(y)$ 必须按 (78) 式下方说明的含义来理解。根据那里的分析可得，处方 (82) 等价于将 $(-\mathcal{B}[\partial/\partial u])$ 直接插在未积分顶点算子 $cV_o(u)$ 的左侧 (或者等价地，将 $\mathcal{B}[\partial/\partial u]$ 直接插在该顶点算子的右侧)。在这种形式下，该处方适用于任意外开弦态，不要求一定在壳。由于插入的算子是格拉斯曼偶算子，关联函数内算子位置不会引发符号歧义。

3. The prescriptions described in items 1 and 2 above hold as long as all the open string vertex operators can be inserted in integrated form and we are allowed to integrate over the sizes of all the boundaries. This holds for $g > 0$ and $g = 0$, $n_c + b \geq 3$, since we can fix the locations of three of the closed string vertex

operators and/or boundaries and integrate over all the open string positions and boundary sizes. However, for $g=0, n_c+b<3$, this is not possible, e.g., for open string amplitudes on the disk, we need to use three unintegrated open string vertex operators.

3. 只要所有开弦顶点算子都可以按积分形式插入，且我们允许对所有边界的大小积分，上述第 1 点和第 2 点描述的处方就成立。这对 $g>0$ 和 $g=0$ 、$n_c+b\geq 3$ 都成立，因为我们可以固定三个闭弦顶点算子和/或边界的位置，然后对所有开弦位置和边界大小积分。但对 $g=0, n_c+b<3$ 而言这无法成立，例如圆盘上的开弦振幅，我们需要使用三个未积分的开弦顶点算子。

The cases to consider here are

我们此处需要考虑的情况是

- $b=1, n_c=0$. These are disk amplitudes with $n_o\geq 3$. Here we shall use the convention that the three unintegrated vertex operators are placed inside the correlator in the order they appear as we travel along the boundary keeping the Riemann surface to the left, and the rest of the open string operators are inserted following the prescription (82). Note that with three Grassmann odd unintegrated vertex operators, there is no further sign ambiguity due to cyclicity.

- $b=1, n_c=0$ 。这些是带有 $n_o\geq 3$ 的盘振幅。此处我们采用如下约定: 当沿边界移动、保持黎曼曲面在左侧时，三个未积分顶点算符按它们出现的顺序置于关联函数内部，其余开弦算符按照规则 (82) 插入。注意，对于三个格拉斯曼奇的未积分顶点算符，cyclicity 不会再引入额外符号歧义。

- $b=1, n_c=1$. For the disk two-point function of one open and one closed strings, there is no ambiguity since the closed string vertex operator is even, and hence the open string vertex operator can be placed anywhere inside the correlator. However, we prescribe that in this case, the correlator comes with an extra sign, i.e., for on-shell vertex operators, we use [22]

- $b=1, n_c=1$ 。对于一开弦一闭弦的盘两点函数，由于闭弦顶点算符是偶的，不存在歧义，因此开弦顶点算符可置于关联函数内任意位置。但我们规定，这种情况下关联函数带有一个额外符号，即对在壳顶点算符，我们采用文献 [22] 的约定

$$\widehat{\Omega}^{(0,1,1,1)}(c\bar{c}V_c; cV_o) = -\langle cV_o c\bar{c}V_c\rangle . \tag{83}$$

Any further open string insertion can be treated using (82). This extra sign is surprising, but it is needed for compatibility with the general prescription for inserting integrated open string vertex operator as given in (82) [22].

任何额外的开弦插入都可以用 (82) 处理。这个额外符号看似意外，但它是为了和 (82) 中给出的积分开弦顶点算符的一般插入规则兼容所必需的 [22]。

- $b=2, n_c=0$. In this case, we shall follow the prescription of replacing the closed string state in the open-closed disk amplitude (83) by the one-form state (81) keeping the open string position unintegrated. Any further open string insertion can be treated using (82).

- $b=2, n_c=0$。这种情况下，我们遵循如下规则：将开-闭盘振幅 (83) 中的闭弦态替换为一元态 (81)，保持开弦位置不积分。任何额外的开弦插入都可以用 (82) 处理。

It has been shown in [22] that with this definition of $\widehat{\Omega}^{(g,b,n_c,n_o)}$, the normalization constants N_{g,b,n_c,n_o} are given by,

文献 [22] 已证明，采用 $\widehat{\Omega}^{(g,b,n_c,n_o)}$ 的这一定义，归一化常数 N_{g,b,n_c,n_o} 由下式给出：

$$N_{g,b,n_c,n_o} = \eta_c^{3g-3+n_c+\frac{3}{2}b+\frac{3}{4}n_o}. \tag{84}$$

where we have defined the constant η_c

其中我们定义了常数 η_c

$$\eta_c \equiv -\frac{1}{2\pi i} = \frac{i}{2\pi}, \tag{85}$$

and have used the freedom of redefining g_s to ensure that the closed string three-point coupling has no extra normalization factor, i.e., $N_{0,0,3,0}=1$.

并且我们利用重新定义 g_s 的自由度，保证闭弦三点耦合不存在额外归一化因子，即 $N_{0,0,3,0}=1$。

We end this subsection with a few comments:

我们在本小节最后给出几点说明：

1. The solution for the N ’ s given in (84) holds for the choice of arrangement of the b ’ s inside the correlator as described earlier in this subsection. Starting from the prescription given above, we can move the b ’ s to wherever we like and/or modify the definition of the orientation of the integral over moduli spaces, at the cost of picking up additional signs. We shall absorb these signs into the definition of $\widehat{\Omega}$ itself, so that (84) is not affected.

1. (84) 中给出的 N 的解适用于本小节前文描述的关联函数内 b 的排布选择。从上述规则出发，我们可以将 b 移到任意位置，和/或修改模空间积分取向的定义，代价是会产生额外符号。我们会将这些符号吸收到 $\widehat{\Omega}$ 本身的定义中，因此 (84) 不受影响。

2. In the specification of the signs, we have worked with $\Omega^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}}$ and taken the arguments A^c_i to be Grassmann even and A^o_i to be Grassmann odd. This will be sufficient to define the string field theory action. However, for intermediate steps in our analysis, we need $\Omega^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}}$ for arguments with wrong Grassmann parity. Since these do not appear in the description of the string field theory action, we can choose them in any way we like and modify the intermediate steps of the analysis accordingly, but for definiteness, we shall adopt the following prescription. Starting with the expression for $\widehat{\Omega}$ given above for even A^c_i ’ s and odd A^o_j ’ s, we first rearrange the operators inside the correlator so that they appear in the order $A^c_1, \cdots A^c_{n_c}, A^o_1, \cdots, A^o_{n_o}$ followed by the b -ghost and boundary state insertions, picking up appropriate signs along the way. We then declare that this formula is valid for arbitrary choice of Grassmann parities of A^c_i ’ s and A^o_j ’ s. Once we

have defined $\widehat{\Omega}^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}}$ for arbitrary Grassmann parities of A^c_i and A^o_j ’ s, the form $\Omega^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}}$ is declared to be given by (69) with N_{g,b,n_c,n_o} given by (84).

2. 在符号约定中，我们针对 $\Omega^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}}$ 开展工作，假设自变量 A^c_i 是格拉斯曼偶的，A^o_i 是格拉斯曼奇的。这足以定义弦场论作用量。但在我们分析的中间步骤中，我们需要对格拉斯曼奇偶性错误的自变量使用 $\Omega^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}}$。由于这类自变量不会出现在弦场论作用量的描述中，我们可以按任意偏好选择它们的定义，并相应修改分析的中间步骤；但为明确起见，我们采用下述规则: 从上述偶 A^c_i、奇 A^o_j 对应的 $\widehat{\Omega}$ 表达式出发，我们首先重新排列关联函数内的算符，使其按照 $A^c_1, \cdots A^c_{n_c}, A^o_1, \cdots, A^o_{n_o}$ 在后接 b 鬼和边界态插入的顺序排列，在此过程中得到相应的符号。随后我们声明，该公式对 A^c_i 和 A^o_j 任意格拉斯曼奇偶性的选择都成立。当我们为 A^c_i 和 A^o_j 的任意格拉斯曼奇偶性定义好 $\widehat{\Omega}^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}}$ 后，我们就规定形式 $\Omega^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}}$ 由式 (69) 给出，其中 N_{g,b,n_c,n_o} 由式 (84) 给出。

3. We also use $\widehat{\Omega}^{(g,b,n_c,n_o)}_p$ for $p \neq d_{g,b,n_c,n_o}$ during the intermediate steps of the analysis since we use identities of type (52). Since these do not appear in the form of the string field theory action, how we define them is up to us. One simple choice is provided by demanding that $\widehat{\Omega}$ satisfies the analog of (52), i.e., we have

3. 由于我们使用 (52) 型恒等式，我们也在分析的中间步骤中为 $p \neq d_{g,b,n_c,n_o}$ 使用 $\widehat{\Omega}^{(g,b,n_c,n_o)}_p$。由于这些不会出现在弦场论作用量的形式中，如何定义它们由我们决定。一个简单的选择是要求 $\widehat{\Omega}$ 满足 (52) 的对应形式，即我们有

$$\widehat{\Omega}^{(g,b,n_c,n_o)}_p \left(QA^c_1, \cdots, A_{n_c}; A^o_1, \cdots, A^o_{n_o}\right) + \cdots$$

$$+(-1)^{A^c_1+\cdots+A^c_n+A^o_1+\cdots A^o_{n_o-1}} \widehat{\Omega}^{(g,b,n_c,n_o)}_p \left(A^c_1, \cdots, A_{n_c}; A^o_1, \cdots, QA^o_{n_o}\right)$$

$$= (-1)^p \, d\widehat{\Omega}^{(g,b,n_c,n_o)}_{p-1} \left(A^c_1, \cdots, A_{n_c}; A^o_1, \cdots, A^o_{n_o}\right), \tag{86}$$

and then declare that $\Omega^{(g,b,n_c,n_o)}_p \left(A^c_1, \cdots, A_{n_c}; A^o_1, \cdots, A^o_{n_o}\right)$ is given by (69) with the same normalization constants N_{g,b,n_c,n_o} as given in (84).

随后声明 $\Omega^{(g,b,n_c,n_o)}_p \left(A^c_1, \cdots, A_{n_c}; A^o_1, \cdots, A^o_{n_o}\right)$ 由式 (69) 给出，其中归一化常数 N_{g,b,n_c,n_o} 与式 (84) 给出的一致。

Batalin-Vilkovisky Formalism

巴塔林-维尔可维斯基形式化

The usual approach to quantizing gauge theories begins with a classical action and its gauge invariance. We then choose some gauge, introduce Faddeev-Popov ghost fields, and carry out the path integral over the original classical fields as well as the ghost fields to compute correlation functions of gauge invariant observables. There is a different but equivalent approach to quantizing gauge theories based on the Batalin-Vilkovisky (BV) formalism [44-46]. While for ordinary gauge theories it gives the same result as the Faddeev-Popov procedure, it has more general applicability and is particularly suited for string field theory. In this subsection, we shall give a very short review of the BV formalism.

规范理论量子化的常规方法从经典作用量及其规范不变性出发。随后我们选定规范，引入法捷耶夫-波波夫鬼场，对原经典场和鬼场做路径积分，来计算规范不变可观测量的关联函数。基于巴塔林-维尔可维斯基 (BV) 形式化，存在另一种等价的规范理论量子化方法 [44-46]。虽然对普通规范理论，它得到的结果与法捷耶夫-波波夫 procedure 一致，但它适用范围更广，尤其适用于弦场论。本小节我们将对 BV 形式化做一个简短综述。

Let us suppose that we have a set of classical fields and some gauge transformations that leave the classical action invariant. Then fields that enter the BV formalism are as follows:

假设我们有一组经典场，以及若干保持经典作用量不变的规范变换，那么进入 BV 形式化的场如下:

1. For every local gauge transformation parameter, we introduce a ghost field that carries Grassmann parity opposite to that of the parameters. Therefore, Grassmann even gauge transformation parameters will lead to Grassmann odd ghost fields and vice versa.

1. 对每一个局域规范变换参数，我们引入一个格拉斯曼奇偶性与该参数相反的鬼场。因此，格拉斯曼偶的规范变换参数对应格拉斯曼奇的鬼场，反之亦然。

2. If the gauge transformation itself has gauge transformations, then we also introduce ghosts for these gauge transformation parameters. The Grassmann parity of these ghosts will be opposite to those of the ghosts associated with the original gauge transformation parameters. For example, if we have a bosonic p -form field Φ_p with gauge invariance under $\Phi_p \to \Phi_p + d\Lambda_{p-1}$ for some $(p-1)$ -form Λ_{p-1} , then we introduce Grassmann odd ghost fields $C^{(1)}_{p-1}$. But the gauge transformation parameter has its own gauge invariance under $\Lambda_{p-1} \to \Lambda_{p-1} + d\chi_{p-2}$ for some $(p-2)$ -form χ_{p-2} . So we need to introduce Grassmann even ghost fields $C^{(2)}_{p-2}$. This will continue until we reach zero-form ghost fields.

2. 如果规范变换本身还存在规范变换，我们也要为这些规范变换参数引入鬼场。这些鬼场的格拉斯曼奇偶性与原规范变换参数对应的鬼场相反。例如，若我们有一个玻色 p 形式场 Φ_p ，它在 $\Phi_p \to \Phi_p + d\Lambda_{p-1}$ 下具有规范不变性，对应某个 $(p-1)$ 形式 Λ_{p-1} ，那么我们引入格拉斯曼奇鬼场 $C^{(1)}_{p-1}$ 。但这个规范变换参数本身在 $\Lambda_{p-1} \to \Lambda_{p-1} + d\chi_{p-2}$ 下还有规范不变性，对应某个 $(p-2)$ 形式 χ_{p-2} 。因此我们需要引入格拉斯曼偶鬼场 $C^{(2)}_{p-2}$ 。这一过程会持续到我们得到零形式鬼场为止。

3. Once we have all the classical fields and the ghost fields, we call them collectively fields. For every field ψ_m , we introduce an anti-field ψ^*_m with opposite Grassmann parity.

3. 得到所有经典场和鬼场后，我们将它们统称为场。对每一个场 ψ_m ，我们引入一个格拉斯曼奇偶性相反的反场 ψ^*_m 。

Given two functionals F and G of the fields and the anti-fields, we define their antibracket as follows:

给定场和反场的两个泛函 F 和 G ，我们定义它们的反括号如下:

$$\{F, G\} = \frac{\partial_r F}{\partial \psi_m} \frac{\partial_\ell G}{\partial \psi^*_m} - \frac{\partial_r F}{\partial \psi^*_m} \frac{\partial_\ell G}{\partial \psi_m}, \tag{87}$$

where the sum over m also includes integration over the space-time coordinates and the subscripts ℓ

and r of ∂ denote left and right derivatives, respectively. Note that left and right derivatives of an object A of Grassmanality $(-1)^A$ are related as follows:

其中对 m 的求和还包含对时空坐标的积分，∂ 的下标 ℓ 和 r 分别表示左导数和右导数。注意格拉斯曼性为 $(-1)^A$ 的对象 A，其左导数和右导数满足下述关系:

$$\frac{\partial_r A}{\partial\psi} = (-1)^{\psi(A+1)} \frac{\partial_l A}{\partial\psi}, \tag{88}$$

with ψ in the exponent equal to one, if ψ is odd and zero if even. The antibracket has a surprising exchange property:

若 ψ 是奇性的，则指数上的 ψ 等于 1，若为偶性则等于 0。反括号具有一个非常特殊的交换性质:

$$\{F, G\} = -(-1)^{(F+1)(G+1)}\{G, F\}. \tag{89}$$

It is symmetric for even entries and antisymmetric for odd ones. We also define a second-order differential operator Δ acting on a functional F of fields and anti-fields:

对偶条目它是对称的，对奇条目它是反对称的。我们还定义一个作用在场和反场的泛函 F 上的二阶微分算子 Δ :

$$\Delta F = (-1)^{\psi_m} \frac{\partial_\ell}{\partial\psi_m} \frac{\partial_\ell F}{\partial\psi_m^*} = (-1)^{\psi_m F} \frac{\partial_r}{\partial\psi_m} \left(\frac{\partial_\ell F}{\partial\psi_m^*}\right). \tag{90}$$

The operator Δ is odd: ΔF and F have opposite Grassmanality. It also has a very important property, $\Delta^2 = 0$, as can be quickly verified with the above formula. In the geometrical description of BV quantization [47], this odd Laplacian Δ squaring to zero and a pointwise multiplication of functions are the starting point for the formulation, with the antibracket a derived concept. We will see how this works in section "BV Structures on Moduli Spaces".

算子 Δ 是奇性的: ΔF 和 F 的格拉斯曼性相反。它还有一个非常重要的性质 $\Delta^2 = 0$，利用上述公式可以快速验证。在 BV 量子化的几何描述 [47] 中，这个平方为零的奇拉普拉斯算子 Δ 和函数的逐点乘法是表述的起点，反括号是导出概念。我们会在“模空间上的 BV 结构”一节讲解具体构造。

The classical BV master action S_{cl} is a functional of the fields and anti-fields that reduces to the original classical action when all the anti-fields are set to zero. It must also satisfy the classical BV master equation,

经典 BV 主作用量 S_{cl} 是场和反场的泛函，当所有反场都置零时，它退化为原经典作用量。它还必须满足经典 BV 主方程,

$$\{S_{cl}, S_{cl}\} = 0 \tag{91}$$

This actually guarantees that the classical master action is invariant under a set of gauge transformations that includes the gauge transformations of the original classical action. The quantum BV master action S satisfies the BV master equation:

这实际上保证了经典主作用量在一组规范变换下不变，这组变换已经包含了原经典作用量的规范变换。量子 BV 主作用量 S 满足 BV 主方程:

$$\frac{1}{2}\{S, S\} + \Delta S = 0, \tag{92}$$

or, equivalently,

或者等价地，

$$\Delta e^S = 0\,. \tag{93}$$

If g_s is the coupling constant of the theory, then tree-level n -point functions are of order g_s^{n-2} , while g -loop n -point functions are of order g_s^{2g+n-2} . Using this, we can see that the term proportional to $\{S, S\}$ dominates the term proportional to ΔS in the limit of small g_s . An action that satisfies the quantum BV equation admits consistent gauge fixing, and the theory has well-defined observables; the S-matrix of the theory is independent of the gauge-fixing conditions.

若 g_s 是该理论的耦合常数，则树图阶 n 点函数是 g_s^{n-2} 阶，而 g 圈 n 点函数是 g_s^{2g+n-2} 阶。由此可知，在 g_s 很小时，正比于 $\{S, S\}$ 的项支配正比于 ΔS 的项。满足量子 BV 方程的作用量允许相容的规范固定，理论拥有良定义的可观测量，理论的 S 矩阵不依赖于规范固定条件。

Gauge fixing in this theory is done as follows. One chooses an arbitrary Grassmann odd functional Ψ of the fields, sets the anti-fields to

该理论的规范固定操作如下。选取任意一个场的格拉斯曼奇泛函 Ψ，将反场设为

$$\psi_m^* = \frac{\partial \ell \Psi}{\partial \psi_m}, \tag{94}$$

and then carries out path integral over the set of fields $\{\psi_m\}$ weighted by e^S for calculation of the correlation functions. A somewhat singular gauge choice is $\Psi = 0$. This sets all the anti-fields to zero, and the master action reduces to the classical action up to corrections that are suppressed by factors of order g_s^2 relative to their classical counterparts. The integration over the classical fields can be interpreted as the original path integral. Since ghosts have opposite statistics compared to the gauge transformation parameters, the integration over the ghost fields can be regarded as division by the volume of the gauge group. This is not suitable, however, for perturbation expansion since the kinetic operators for gauge fields are not invertible, and therefore, propagators cannot be defined. More familiar choices of gauge correspond to other choices of Ψ .

随后对场组态 $\{\psi_m\}$ 做带权重 e^S 的路径积分来计算关联函数。一个较为特殊的规范选择是 $\Psi = 0$，该选择将所有反场都置零，主作用量退化为经典作用量，修正项相比经典项被 g_s^2 阶因子压低。对经典场的积分可以解释为原本的路径积分。由于鬼的统计性质与规范变换参数相反，对鬼场的积分可以看作除以规范群的体积。但该方法不适用于微扰展开，因为规范场的动能算子不可逆，因此无法定义传播子。更常用的规范选择对应 Ψ 的其他选取。

Even though we introduced the anti-fields in a way that seems to put them on a different footing than the fields, we can make field redefinitions that mix the fields and anti-fields preserving the antibracket, i.e., the antibrackets computed using the new fields and anti-fields must give us back the original result. For example, it is easy to check that with new tilde fields and tilde anti-fields given by

尽管我们引入反场的方式看似让反场与场处于不同地位，但我们可以做场重定义，将场与反场混合同时保持反括号不变，也就是说，用新的场和反场计算得到的反括号必须给出和原结果相同的结果。例如，不难验证，当新的带波浪线的场和带波浪线的反场由下式给出时

$$\widetilde{\psi}_m^* = \psi_m^* - \frac{\partial_\ell \Psi}{\partial \psi_m},\ \widetilde{\psi}_m = \psi_m, \tag{95}$$

the antibracket is left invariant. Therefore, the gauge fixing described in (94) can also be regarded as setting the new anti-fields to zero. To make this freedom of redefining the fields and anti-fields manifest, one often combines the fields and the anti-fields into a single set of fields ψ^i and introduces a symplectic pairing ω^{ij} between fields of opposite Grassmann parity, in terms of which we can write,

反括号保持不变。因此，条目 (94) 中描述的规范固定也可以看作将新反场置零。为了明确体现这种场和反场重定义的自由度，通常将场和反场合并为单一的场集合 ψ^i ，并在格拉斯曼奇偶性相反的场之间引入辛配对 ω^{ij} ，利用它我们可以写出

$$\{F, G\} = \frac{\partial_r F}{\partial \psi^i} \omega^{ij} \frac{\partial_l G}{\partial \psi^j},\ \omega^{ij} = -\omega^{ji},$$

$$\Delta F = \frac{1}{2}(-1)^{\psi^i} \frac{\partial_\ell}{\partial \psi^i}\left(\omega^{ij} \frac{\partial_\ell F}{\partial \psi^j}\right) = \frac{1}{2}(-1)^{\psi^i F} \frac{\partial_r}{\partial \psi^i}\left(\omega^{ij} \frac{\partial_\ell F}{\partial \psi^j}\right). \tag{96}$$

Bosonic and Superstring Field Theories

玻色弦场论与超弦场论

The main goal of this section is to give a quick overview of all the bosonic and superstring field theories. The detailed mathematical structure behind these constructions will be described in the later sections. We shall first describe various closed string field theories (bosonic, type II, and heterotic) and then turn to tree-level open string field theory and open-closed string field theory. We write explicitly the kinetic terms for the massless sector of the free bosonic string field theories. The free string field theories require the BRST operator of the conformal field theory and the BPZ inner product reviewed in the previous section, which is supplemented with a c_0^- insertion for the closed string theory. Such free field theories were first considered in the early work on covariant string theory [48-51]. Some of the early approaches to covariant string field theory used the light-cone version of the string vertices, with an auxiliary string length parameter [52-54]. This Kyoto group formulation was consistent for the classical theories but had complications at the quantum level. [6]

本节的主要目标是快速概述所有玻色弦场论与超弦场论, 这些构造背后的详细数学结构将在后续章节介绍。我们首先描述各类闭弦场论 (玻色、II 型和杂化弦), 随后讨论树级开弦场论与开-闭弦场论。我们明确写出了自由玻色弦场论无质量 sector 的动能项。自由弦场论需要用到共形场论的 BRST 算符和上一节回顾的 BPZ 内积，闭弦理论还需要额外插入一个 c_0^- 。这类自由场论最早出现在协变弦论的早期研究中 [48-51]。早期协变弦论的部分方法使用带辅助弦长参数的光锥版本弦顶点 [52-54]。这种京都学派的 formulation 对经典理论是自洽的，但在量子层面存在复杂性。[6]

We begin with preliminary comments on the string field. The original ingredient is the (complex) vector space of a CFT or some well-defined subspace of this vector space. We call this vector space $\mathcal{H}$ and view it as the space spanned by a set of local vertex operators. In this vector space, there is a grading, a degree usually related to ghost number that determines the Grassmanality of the operators. Let the vertex operator φ_i be a basis vector in $\mathcal{H}$ of definite Grassmanality. For every such basis vector, we introduce a target-space field ψ^i with Grassmanality correlated to that of the operator φ_i , opposite Grassmanality for open strings, the same Grassmanality for closed strings. The string field vertex operator Ψ is obtained by summing over all basis vectors the product of the basis vector times the associated target-space field:

我们首先对弦场做预备说明。原始构成是共形场论的 (复) 向量空间，或是该向量空间的某个良定义子空间。我们将这个向量空间称为 $\mathcal{H}$ ，并将其视作由一组局域顶点算子张成的空间。这个向量空间存在分次，其次数通常和鬼数相关，决定了算子的格拉斯曼性。设顶点算子 φ_i 是 $\mathcal{H}$ 中具有确定格拉斯曼性的基向量。对每个这样的基向量，我们引入目标场 ψ^i ，其格拉斯曼性与算子 φ_i 相关: 开弦取相反格拉斯曼性，闭弦取相同格拉斯曼性。弦场顶点算子 Ψ 可通过对所有基向量求和得到, 即基向量乘以对应目标空间场的乘积之和:

$$\Psi = \sum_i \varphi_i \psi^i \tag{97}$$

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[6] An approach to a covariantized light-cone version with a physical string length was discussed by T. Kugo (see [55]).

[6] T. Kugo 讨论了一种带物理弦长的协变化光锥版本方案 (见 [55])。

The Grassmanality of each term in the above sum is the sum of the Grassmanality of the basis vector φ_i and that of the target space field ψ^i . Given the above stated correlations, the bosonic open string field vertex operator is odd, and the bosonic closed string field vertex operator is even. With the target space fields ψ^i being Grassmann objects (and not plain complex numbers), string fields are not just elements of the complex vector space $\mathcal{H}$ but rather elements of a G -module $\mathcal{H}_G$, where the G is for Grassmann [56]. The string field action itself is an element of the Grassmann algebra, a function of the target space fields. Having noted this, in what follows, we will identify (with a little abuse of notation) the vector space $\mathcal{H}$ with the module $\mathcal{H}_G$. One consequence of this is that the Grassmanality of an element of $\mathcal{H}$ is no longer determined only by its ghost number.

上述和式中每一项的格拉斯曼性等于基向量 φ_i 的格拉斯曼性加上目标空间场 ψ^i 的格拉斯曼性。根据上述关联规则，玻色开弦的弦场顶点算子是奇性的，玻色闭弦的弦场顶点算子是偶性的。由于目标空间场 ψ^i 是格拉斯曼对象 (而非普通复数)，弦场不只是复向量空间 $\mathcal{H}$ 的元素，而是 G 模 $\mathcal{H}_G$ 的元素，其中 G 对应格拉斯曼代数 [56]。弦场作用量本身是格拉斯曼代数的元素，是目标空间场的函数。明确这一点后，在下文中我们将 (稍作记号滥用) 把向量空间 $\mathcal{H}$ 等同于模 $\mathcal{H}_G$ 。这样做的一个结果是，$\mathcal{H}$ 中元素的格拉斯曼性不再仅由其鬼数决定。

It is sometimes convenient to work with a string field state $|\Psi\rangle = \Psi(0)|0\rangle$, obtained by letting the string field vertex operator act on the vacuum. Writing $|\varphi_i\rangle = \varphi_i(0)|0\rangle$ and letting Eq. (97) act on the vacuum, we have

使用弦场态 $|\Psi\rangle = \Psi(0)|0\rangle$ 有时会更方便，它由弦场顶点算子作用在真空上得到。写出 $|\varphi_i\rangle = \varphi_i(0)|0\rangle$ 并将式 (97) 作用在真空上，我们得到

$$|\Psi\rangle = \sum_i (-1)^{\psi^i \varepsilon_0} |\varphi_i\rangle \psi^i, \tag{98}$$

where ψ^i in the exponent is even (odd) if ψ^i is Grassmann even (odd) and ε_0 is even (odd) if $|0\rangle$ is Grassmann even (odd). Note that when the Grassmanality of the vacuum is even, both Ψ and $|\Psi\rangle$ have the same Grassmanality. When the Grassmanality of the vacuum is odd, Ψ and $|\Psi\rangle$ have opposite Grassmanality. As we will see, it is sometimes useful for open strings to use a Grassmann odd vacuum. For closed strings, the Grassmanality of the vacuum is chosen to be even, so both the vertex operator and state picture of the string field have the same Grassmanality, and the sign factor in the above equation is not needed. When stating the Grassmanality of the string field below, we are referring to the vertex operator string field, unless noted otherwise.

其中若 $|0\rangle$ 是格拉斯曼偶 (奇) 的，则指数中的 ψ^i 为偶 (奇)，ε_0 为偶 (奇)。注意，当真空格拉斯曼性为偶时，Ψ 与 $|\Psi\rangle$ 的格拉斯曼性相同；当真空格拉斯曼性为奇时，Ψ 与 $|\Psi\rangle$ 的格拉斯曼性相反。我们下文会提到，对开弦使用格拉斯曼奇真空有时会更方便。对闭弦，真空的格拉斯曼性通常取偶，因此弦场的顶点算子图景与态图景具有相同格拉斯曼性，上述等式中的符号因子不需要存在。下文提及弦场的格拉斯曼性时，除非另有说明，均指顶点算子形式的弦场。

Closed Bosonic String Field Theory

闭玻色弦场论

We begin with the closed bosonic string field theory, mostly following [6] and references cited there, with the perspective that later developments have given. The formulation of closed bosonic string field theory involves the following ingredients:

我们从闭玻色弦场论开始，主要参考 [6] 及其中引用文献，并结合后续发展所得的视角阐述。闭玻色弦场论的构造包含以下要素:

1. A string field $|\Psi\rangle$ taking value in the space of states of the world-sheet conformal | field theory of matter and ghost fields, satisfying the level-matching conditions

1. 满足能级匹配条件的弦场 $|\Psi\rangle$ taking value in the space of states of the world-sheet conformal | , 即物质场与鬼场的场论

$$b_0^- \,|\Psi\rangle = 0, \; L_0^- |\,\Psi\rangle = 0. \tag{99}$$

We have called $\mathcal{H}_c$ this subspace of the world-sheet conformal field theory, and thus we have

我们已经将世界面共形场论的这个子空间称为 $\mathcal{H}_c$ ，因此我们得到

$$\Psi \in \mathcal{H}_c \,. \tag{100}$$

As discussed in section ”World-Sheet Conventions”, the natural non-degenerate bilinear inner product $\langle A, B\rangle$ in $\mathcal{H}_c$ is given by (15)

如“世界面约定”一节所述，$\mathcal{H}_c$ 中自然的非退化双线性内积 $\langle A, B\rangle$ 由式 (15) 给出

$$\langle A, B\rangle \equiv \langle A\,|c_0^-|\,B\rangle \;\; |A\rangle, |B\rangle \in \mathcal{H}_c. \tag{101}$$

and satisfies

且满足

$$\langle A, B\rangle = (-1)^{(A+1)(B+1)} \langle B, A\rangle, \tag{102}$$

as well as

以及

$$\langle \lambda A, B\eta\rangle = \lambda \langle A, B\rangle \eta, \tag{103}$$

for λ, η constants of arbitrary Grassmanality.

对任意格拉斯曼性的 λ, η 常数成立。

2. The string field vertex operator Ψ is Grassmann even, i.e., if ζ is a Grassmann odd number, then $\zeta\Psi = \Psi\zeta$. The state $|\Psi\rangle$ is also even. If we expand Ψ as a linear combination of vertex operators, then the coefficient of vertex operators of even ghost number must be Grassmann even, and the coefficient of the vertex operator of odd ghost number must be Grassmann odd.

2. 弦场顶点算符 Ψ 是格拉斯曼偶的，即若 ζ 是格拉斯曼奇数，则 $\zeta\Psi = \Psi\zeta$ 。态 $|\Psi\rangle$ 同样是偶的。若我们将 Ψ 展开为顶点算符的线性组合，则偶鬼数顶点算符的系数必须是格拉斯曼偶的，奇鬼数顶点算符的系数必须是格拉斯曼奇的。

3. The background independent geometric content of the theory is encapsulated by string "vertices" $\mathcal{V}_{g,n}$, which are $6g-6+2n$ -dimensional chains in $\widehat{\mathcal{P}}_{g,n}$, symmetric under the permutations of the punctures and satisfying the geometric form of the BV master equation:

3. 该理论不依赖背景的几何内容封装在弦“顶点”$\mathcal{V}_{g,n}$ 中，它们是 $\widehat{\mathcal{P}}_{g,n}$ 中的 $6g-6+2n$ 维链，在穿孔置换下对称，且满足 BV 主方程的几何形式:

$$\partial \mathcal{V}_{g,n} = -\Delta_c \mathcal{V}_{g-1,n+2} - \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+2}} \left\{ \mathcal{V}_{g_1,n_1}, \mathcal{V}_{g_2,n_2} \right\}_c. \tag{104}$$

We shall explain the various terms in this equation below. First of all, the $\mathcal{V}_{g,n}$ exist for the following values of g and n :

我们接下来会解释这个方程中的各项。首先，$\mathcal{V}_{g,n}$ 存在于 g 和 n 取以下值时:

$$\text{General collection of } \mathcal{V}_{g,n} = \begin{cases} \mathcal{V}_{0,n}, & n \geq 3, \\ \mathcal{V}_{1,n}, & n \geq 1, \\ \mathcal{V}_{g,n}, & n \geq 0, g \geq 2. \end{cases} \tag{105}$$

Of course, it may be that the $\mathcal{V}_{g,n}$ are parts of sections over $\widehat{\mathcal{P}}_{g,n}$ or, if not, subspaces of $\widehat{\mathcal{P}}_{g,n}$ mapping with degree one to the image on $\mathcal{M}_{g,n}$. But the general case is when the $\mathcal{V}_{g,n}$ are chains, which, of course, include the simpler possibilities. The Δ_c and $\{,\}_c$ operations on the chains in $\widehat{\mathcal{P}}_{g,n}$ are defined as follows. First, let us consider $\left\{ \mathcal{V}_{g_1,n_1}, \mathcal{V}_{g_2,n_2} \right\}_c$. Let us pick Riemann surfaces $\sum_{g_1,n_1}$ and $\sum_{g_2,n_2}$, equipped with local coordinates at the punctures, corresponding to particular points in $\mathcal{V}_{g_1,n_1}$ and $\mathcal{V}_{g_2,n_2}$, respectively. We now pick one puncture of $\sum_{g_1,n_1}$ and another puncture of $\sum_{g_2,n_2}$ and call their local coordinates w_1 and w_2 , respectively. Due to the symmetry of the vertices under the permutation of the punctures, it does not matter which punctures we choose. We now glue the Riemann surfaces $\sum_{g_1,n_1}$ and $\sum_{g_2,n_2}$ by identifying the local coordinates w_1 and w_2 via

当然，$\mathcal{V}_{g,n}$ 可能是 $\widehat{\mathcal{P}}_{g,n}$ 上截面的一部分，若非如此，则是映射到 $\mathcal{M}_{g,n}$ 上像的次数为 1 的 $\widehat{\mathcal{P}}_{g,n}$ 子空间。但一般情况 $\mathcal{V}_{g,n}$ 是链，上述更简单的情形自然也包含在其中。$\widehat{\mathcal{P}}_{g,n}$ 中链上的 Δ_c 与 $\{,\}_c$ 运算定义如下。首先我们考虑 $\{\mathcal{V}_{g_1,n_1}, \mathcal{V}_{g_2,n_2}\}_c$ 。取两个黎曼曲面 $\sum_{g_1,n_1}$ 和 $\sum_{g_2,n_2}$ ，它们在穿孔处都配备了局部坐标，分别对应 $\mathcal{V}_{g_1,n_1}$ 和 $\mathcal{V}_{g_2,n_2}$ 中的特定点。现在我们选取 $\sum_{g_1,n_1}$ 的一个穿孔和 $\sum_{g_2,n_2}$ 的另一个穿孔，分别将它们的局部坐标记为 w_1 和 w_2 。由于顶点在穿孔置换下具有对称性，我们选择哪两个穿孔并不影响结果。现在我们通过将局部坐标 w_1 和 w_2 等同，粘合黎曼曲面 $\sum_{g_1,n_1}$ 和 $\sum_{g_2,n_2}$ ，方式如下:

$$w_1 w_2 = e^{i\theta}, \ 0 \leq \theta \leq 2\pi. \tag{106}$$

This twist gluing operation generates a one-parameter family of Riemann surfaces of genus $g_1 + g_2$ and n_1+n_2-2 punctures, labelled by θ . Furthermore, the local coordinates at the punctures on $\sum_{g_1,n_1}$ and $\sum_{g_2,n_2}$ give local coordinates at the $(n_1 + n_2 - 2)$ punctures of the new Riemann surfaces. Repeating this process for each pair of points in $\mathcal{V}_{g_1,n_1}$ and $\mathcal{V}_{g_2,n_2}$, we generate a $(6g_1 - 6 + 2n_1) + (6g_2 - 6 + 2n_2) + 1$ -dimensional chain in $\widetilde{\mathcal{P}}_{g_1+g_2,n_1+n_2-2}$. This chain is defined to be $\left\{ \mathcal{V}_{g_1,n_1}, \mathcal{V}_{g_2,n_2} \right\}_c$.

这种扭转粘合操作会生成一个由 θ 标记的、亏格为 g_1+g_2 且带有 n_1+n_2-2 个孔的单参数黎曼曲面族。此外，$\sum_{g_1,n_1}$ 和 $\sum_{g_2,n_2}$ 上孔处的局部坐标给出了新黎曼曲面 (n_1+n_2-2) 个孔处的局部坐标。对 $\mathcal{V}_{g_1,n_1}$ 和 $\mathcal{V}_{g_2,n_2}$ 中的每对点重复该过程，我们就得到了 $\widehat{\mathcal{P}}_{g_1+g_2,n_1+n_2-2}$ 中的一个 $(6g_1-6+\ 2n_1)+(6g_2-6+2n_2)+1$ 维链。这个链被定义为 $\{\mathcal{V}_{g_1,n_1},\mathcal{V}_{g_2,n_2}\}_c$。

The definition of $\Delta_c\mathcal{V}_{g,n}$ is similar; we pick a surface $\sum_{g,n}$ corresponding to a point in $\mathcal{V}_{g,n}$ and identify the local coordinates of two punctures on $\sum_{g,n}$ as in (106). This is repeated for every surface in $\mathcal{V}_{g,n}$. As a result, the Δ_c operation increases the genus by one and reduces the number of punctures by two, producing a $6g-6+2n+1$ -dimensional chain of $\widehat{\mathcal{P}}_{g+1,n-2}$.

$\Delta_c\mathcal{V}_{g,n}$ 的定义类似；我们选取一个对应 $\mathcal{V}_{g,n}$ 中某点的曲面 $\sum_{g,n}$，并按照式 (106) 的方式标识 $\sum_{g,n}$ 上两个孔的局部坐标。对 $\mathcal{V}_{g,n}$ 中的每个曲面都重复这一步骤。最终，该 Δ_c 操作使亏格增加 1，孔的数量减少 2，得到了 $\widehat{\mathcal{P}}_{g+1,n-2}$ 的一个 $6g-6+2n+1$ 维链。

4. We define a set of multilinear maps from $\mathcal{H}_c^{\otimes n}$ to the space of complex numbers or, more precisely when using string fields, to the Grassmann algebra of target space fields. The maps, denoted as $\{A_1,\cdots,A_n\}$, with $n\geq 1$ are defined as follows: [7]

4. 我们定义一组从 $\mathcal{H}_c^{\otimes n}$ 到复数域的多重线性映射，更准确地说，当使用弦场时，是到目标空间场的格拉斯曼代数的映射。这些映射记为 $\{A_1,\cdots,A_n\}$，其中 $n\geq 1$ 定义如下: [7]

$$\{A_1,\cdots,A_n\}\equiv\sum_{g=0}^{\infty}(g_s)^{-\chi_{g,n}}\{A_1,\cdots,A_n\}_g$$

(107)

$$\equiv\sum_{g=0}^{\infty}g_s^{2g+n-2}\int_{\mathcal{V}_{g,n}}\Omega^{(g,n)}_{d_{g,n}}(A_1,\cdots,A_n).$$

Here, $\mathcal{V}_{g,n}$ are the string vertices introduced above, and $\Omega_p^{(g,n)}$ is the p -form in $\widehat{\mathcal{P}}_{g,n}$ defined in (44). For $n=1,2$, the above sum over genus begins at $g=1$. The products $\{A_1,\cdots,A_n\}$ are graded commutative.

此处，$\mathcal{V}_{g,n}$ 是上文引入的弦顶点，$\Omega_p^{(g,n)}$ 是式 (44) 中定义的 $\widehat{\mathcal{P}}_{g,n}$ 上的 p -形式。对于 $n=1,2$，上述对亏格的求和从 $g=1$ 开始。乘积 $\{A_1,\cdots,A_n\}$ 是阶化交换的。

5. We also define multilinear string products $[A_1,\cdots,A_n]$ that map $\mathcal{H}_c^{\otimes n}$ to $\mathcal{H}_c$, using the multilinear maps to the complex numbers and the bilinear inner product:

5. 我们还利用到复数域的多重线性映射和双线性内积，定义了多重线性弦乘积 $[A_1,\cdots,A_n]$，它将 $\mathcal{H}_c^{\otimes n}$ 映射到 $\mathcal{H}_c$:

$$\langle A_0,[A_1,\cdots,A_n]\rangle=\{A_0,\cdots,A_n\},\ \forall A_0\in\mathcal{H}_c. \tag{108}$$

[7] We are using the same symbol $\{\cdots\}$ to denote the multilinear map from product of $\mathcal{H}_c$ to the Grassmann algebra and antibrackets. Which one we are using should be clear from the context.

[7] 我们使用同一个符号 $\{\cdots\}$ 来表示从 $\mathcal{H}_c$ 的乘积到格拉斯曼代数的多重线性映射和反括号，具体含义可由上下文区分。

Just like the multilinear maps to the complex numbers are sums of contributions over genus, we have

正如到复数域的多重线性映射是各亏格贡献的求和，我们有

$$[A_1,\cdots,A_n] \equiv \sum_{g=0}^{\infty} g_s^{2g+n-1}[A_1,\cdots,A_n]_g. \tag{109}$$

Due to the Grassmann odd c_0^- factor in the definition of the inner product given in (15), $[A_1,\cdots,A_n]$ has opposite Grassmann parity compared to the sum of the Grassmann parities of the A_i ’ s (the product carries intrinsic degree one). At a practical level, we can implement this by declaring that while moving a Grassmann odd number through [we pick an extra minus sign, i.e., [acts as a Grassmann odd object. The other property that will be useful is that if $A_1,\cdots,A_n$ are Grassmann even, then $[A_1,\cdots,A_n]$ is Grassmann odd and $\{A_1,\cdots,A_n\}$ is Grassmann even.

由于 (15) 给出的内积定义中存在格拉斯曼奇因子 c_0^- ，与所有 A_i 的格拉斯曼奇偶性之和相比，$[A_1,\cdots,A_n]$ 具有相反的格拉斯曼奇偶性 (乘积本身带一次内禀次数)。实际操作中，我们可以通过下述方式实现这一点: 当移动一个格拉斯曼奇数穿过 [时，我们会额外得到一个负号，即 [的作用相当于一个格拉斯曼奇数对象。另一个有用的性质是: 若 $A_1,\cdots,A_n$ 均为格拉斯曼偶，则 $[A_1,\cdots,A_n]$ 为格拉斯曼奇，$\{A_1,\cdots,A_n\}$ 为格拉斯曼偶。

6. As a consequence of the geometric recursion (104) and the CFT Ward identities (52), we have the main identity [6],

6. 由几何递推关系 (104) 和共形场论 Ward 恒等式 (52)，我们得到主恒等式 [6]:

$$\begin{aligned} &\sum_{i=1}^{N}\left\{A_1' \ldots A_{i-1}'\left(QA_i'\right)A_{i+1}' \ldots A_N'\right\} \\ &= -\frac{1}{2}\sum_{\substack{\ell,k\geq 0\\ \ell+k=N}}\ \sum_{\substack{\{i_a;a=1,\ldots\ell\}\\ \{j_b;b=1,\ldots k\}}}\left\{A_{i_1}' \ldots A_{i_\ell}'\varphi_s\right\}\left\{\varphi_r A_{j_1}' \ldots A_{j_k}'\right\}\langle\varphi_s^c,\varphi_r^c\rangle \\ &\qquad \{j_b\} := 1,\ldots N\} \\ &\quad -\frac{1}{2}\left\{A_1' \ldots A_N'\varphi_s\varphi_r\right\}\langle\varphi_s^c,\varphi_r^c\rangle, \end{aligned} \tag{110}$$

where $A_i' = \varepsilon_i A_i$ (without any sum over i) where ε_i is a Grassmann even (odd) c-number if A_i is Grassmann even (odd). Therefore, the A_i' ’ s are always Grassmann even. The sum over i_k ’ s and j_k ’ s runs over inequivalent sets, e.g., we do not sum over different permutations of i_k ’ s or different permutations of j_k ’ s. However, in the second line, we do sum over the sets related by the exchange of $\{i_1,\cdots,i_\ell\}$ and $\{j_1,\cdots,j_k\}$

-this is cancelled by the factor of 1/2 multiplying the sum. The states $\{|\varphi_r\rangle\}$ and $\{|\varphi_r^c\rangle\}$ are both complete set of basis states in $\mathcal{H}_c$, satisfying,

其中 $A_i' = \varepsilon_i A_i$ (不对 i 求和)，若 A_i 是格拉斯曼偶 (奇)，则 ε_i 是格拉斯曼偶 (奇)c 数。因此，所有 A_i' 始终为格拉斯曼偶。对 i_k 和 j_k 的求和遍历不等价集合，例如我们不对 i_k 的不同排列或 j_k 的不同排列重复求和。但在第二行中，我们对交换 $\{i_1, \cdots, i_\ell\}$ 与 $\{j_1, \cdots, j_k\}$ 得到的集合都进行求和——这会被乘在求和项前的因子 1/2 抵消。态 $\{|\varphi_r\rangle\}$ 和 $\{|\varphi_r^c\rangle\}$ 都是 $\mathcal{H}_c$ 中的完备基，满足:

$$\langle \varphi_r^c, \varphi_s\rangle = \langle \varphi_s, \varphi_r^c\rangle = \delta_{rs}. \tag{111}$$

The first equality follows because for any non-vanishing inner product of vertex operators, the two operators must have opposite Grassmanality, and therefore, symmetry follows from the exchange relation (102). Therefore, we also have

第一个等号成立是因为: 对于顶点算符的任何非零内积, 这两个算符必须具有相反的格拉斯曼性, 因此对称性可由交换关系 (102) 推出。由此我们还得到

$$\left\langle \varphi_i, \varphi_j\right\rangle = \left\langle \varphi_j, \varphi_i\right\rangle, \text{ and } \left\langle \varphi_i^c, \varphi_j^c\right\rangle = \left\langle \varphi_j^c, \varphi_i^c\right\rangle. \tag{112}$$

Note also that any non-vanishing overlap is necessarily a commuting number. Of course, the two basis vectors are related, for example, $\varphi_j = \sum_k \left\langle \varphi_j, \varphi_k\right\rangle \varphi_k^c$.

还需注意，任何非零重叠必然是对易数。当然，这两组基矢是相关联的，例如 $\varphi_j = \sum_k \left\langle \varphi_j, \varphi_k\right\rangle \varphi_k^c$ 。

We shall not give a detailed proof of (110), but the logic behind the proof can be found in section "Geometric BV Master Equation and String Field Theory Master Equation", where we shall give a direct proof of the fact that the string field theory action satisfies the BV master equation.

我们不会给出 (110) 的详细证明，但证明的逻辑可以在“几何 BV 主方程与弦场论主方程”一节中找到，在那里我们会直接证明弦场论作用量满足 BV 主方程这一结论。

We can strip off the products of ε_i ’ s from both sides of (110) by moving them to the extreme left using the properties mentioned below (109) and write the identity in terms of the original variables A_i by keeping track of the signs of various terms that arise while making the rearrangement of the ε_i ’ s. Note that this main identity, with N arbitrary but fixed, contains a number of identities relating multilinear products at various genera. These are obtained by expansion of all terms in powers of g_s , with independent equalities holding for each power of g_s .

我们可以利用 (109) 下方提到的性质，将 ε_i 的乘积移到最左侧，从 (110) 的两侧消去这些乘积，再追踪重排 ε_i 过程中各项产生的符号，用原始变量 A_i 写出该恒等式。注意这个主恒等式在 N 任意但固定的情况下，包含了多个联系不同亏格多重线性乘积的恒等式。这些恒等式可以通过将所有项按 g_s 的幂次展开得到，g_s 的每个幂次都对应独立的等式。

7. The main identity (110) given above can also be expressed in terms of the products [ ] as follows:

7. 上述主恒等式 (110) 也可以用乘积 [ ] 表示如下:

$$Q\left[A_1' \cdots A_N'\right] = -\sum_{i=1}^{N}\left[A_1' \ldots A_{i-1}'\left(QA_i'\right)A_{i+1}' \ldots A_N'\right] \tag{113}$$

$$-\sum_{\substack{\ell,k\geq 0\\ \ell+k=N}} \sum_{\substack{\{i_a;a=1,\ldots\ell\}\\ \{j_b;b=1,\ldots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\ldots N\}}} \left[A_{i_1}' \ldots A_{i_\ell}'\left[A_{j_1}' \ldots A_{j_k}'\right]\right]$$

$$-\frac{1}{2}\left[A_1' \ldots A_N'\varphi_s\varphi_r\right]\langle\varphi_s^c, \varphi_r^c\rangle .$$

As we shall explain in section " L_∞ Algebras and Classical Closed String Field Theory",(113) without the last term defines an L_∞ algebra. With the last term, it is a "quantum" version of the L_∞ algebra.

正如我们将在“L_∞ 代数与经典闭弦场论”一节 (113) 中解释的，去掉最后一项后定义了一个 L_∞ 代数。加上最后一项，它就是 L_∞ 代数的“量子”版本。

8. The BV master action of closed bosonic string field theory is given by,

8. 闭玻色弦场论的 BV 主作用量由下式给出,

$$S = \frac{1}{2}\langle\Psi, Q\Psi\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\{\Psi^n\} \tag{114}$$

As will be explained in (438), this action satisfies the BV master equation (92) with the antibracket and Δ defined as in (96), with the following definition of ω^{ij} . Let us expand the string field Ψ in the complete set of basis states φ_r as

正如将在 (438) 中解释的，该作用量满足 BV 主方程 (92)，其中反括号与 Δ 按 (96) 定义，并带有如下 ω^{ij} 的定义。我们将弦场 Ψ 按完备基矢组 φ_r 展开为

$$\Psi = \sum_r \varphi_r\psi^r \to \delta\Psi = \sum_r \varphi_r\delta\psi^r \tag{115}$$

where we noted the expression for a first-order variation of the string field. Let F and G be two functions of the string field Ψ , i.e., functions of the coefficients ψ^r of expansion of Ψ are some basis in $\mathcal{H}_c$. We use their variations to define string fields F_R and G_L :

其中我们给出了弦场一阶变分的表达式。设 F 和 G 是弦场 Ψ 的两个函数，即 Ψ 展开系数 ψ^r 的函数，而 ψ^r 是 $\mathcal{H}_c$ 中的某个基。我们利用它们的变分定义弦场 F_R 和 G_L :

$$\delta F = \sum_k \frac{\partial^r F}{\partial\psi^k}\delta\psi^k = \langle F_R, \delta\Psi\rangle \ \to \ \frac{\partial^r F}{\partial\psi^k} = \langle F_R, \varphi_k\rangle$$

$$\delta G = \sum_k \delta\psi^k\frac{\partial^l G}{\partial\psi^k} = \langle\delta\Psi, G_L\rangle \ \to \ \frac{\partial^l G}{\partial\psi^k} = (-1)^{\varphi_k}\langle\varphi_k, G_L\rangle , \tag{116}$$

where φ_k in the exponent denotes the ghost number of φ_k . Then we define the antibracket of F and G to be

其中指数上的 φ_k 表示 φ_k 的鬼数。随后我们将 F 和 G 的反括号定义为

$$\{F, G\} = -\langle F_R, G_L \rangle . \tag{117}$$

Then we have, comparing (96) and (117),

对比 (96) 和 (117)，我们得到

$$\omega^{rs} = (-1)^{\varphi_s + 1} \langle \varphi_r^c, \varphi_s^c \rangle , \; \omega_{rs} = (-1)^{\varphi_r + 1} \langle \varphi_r, \varphi_s \rangle , \tag{118}$$

where ω_{rs} is the matrix inverse of ω^{rs} . Using (118), one can check that among an infinite number of choices, the following decompositions into fields and anti-fields are possible and particularly useful:

其中 ω_{rs} 是 ω^{rs} 的逆矩阵。利用 (118) 可以验证，在无穷多种选择中，如下分解为场和反场的方案是可行且尤为有用的:

(a) Up to signs, the coefficients of the states of ghost number ≥ 3 can be regarded as anti-fields, and the coefficients of the states of ghost number ≤ 2 can be regarded as fields.

(a) 不计符号，鬼数为 ≥ 3 的态的系数可视为反场，鬼数为 ≤ 2 的态的系数可视为场。

(b) Up to signs, the coefficients of the states annihilated by c_0^+ can be regarded as anti-fields, and the coefficients of the states annihilated by b_0^+ can be regarded as fields.

(b) 不计符号，被 c_0^+ 零化的态的系数可视为反场，被 b_0^+ 零化的态的系数可视为场。

Note that, as required, for such choices, ω_{rs} does not couple fields to fields, nor does it couple anti-fields with anti-fields.

请注意，按要求，对于这类选择，ω_{rs} 不将场与场耦合，也不将反场与反场耦合。

Requiring that the variation of the string action (114) under first-order variation of Ψ vanishes and using the non-degeneracy of the inner product, we get the string field equation. This can be expressed in terms of string products as follows:

要求弦作用量 (114) 在 Ψ 的一阶变分下变分为零，并利用内积的非退化性，我们得到弦场方程。这可以用弦乘积表述如下:

$$Q\Psi + \sum_{n=1}^{\infty} \frac{1}{n!} [\Psi^n] = 0. \tag{119}$$

If in the definition of $\{\cdots\}$ and $[\cdots]$ we include only the genus zero contribution to the correlators, then they will satisfy an identity of the form (110) (or (113)) with only genus zero contributions and without a contribution from the last term in the identity. Indeed, at genus zero, the multilinear form has the power g_s^{N-2} with N the number of string fields. Therefore, the first term in (110) has a power of g_s^{N-2} , and the term in the second line has a power of $g_s^{\ell+1-2} g_s^{k+1-2} = g_s^{\ell+k-2} = g_s^{N-2}$, but the last term has a power $g_s^{N+2-2} = g_s^N$, so it does not contribute to the relation. The action S , restricted also to genus zero contribution, is the classical master action

> 如果在 $\{\cdots\}$ 和 $[\cdots]$ 的定义中我们仅包含关联函数的亏格零贡献,那么它们将满足形如 (110)(或 (113)) 的恒等式，其中仅含亏格零贡献，且不包含恒等式中最后一项的贡献。实际上，在亏格零时，多重线性形式的幂次为 g_s^{N-2} ，其中 N 是弦场的个数。因此，(110) 中的第一项幂次为 g_s^{N-2} ，第二行的项幂次为 $g_s^{\ell+1-2} g_s^{k+1-2} = g_s^{\ell+k-2} = g_s^{N-2}$ ，而最后一项的幂次为 $g_s^{N+2-2} = g_s^N$ ，因此它对该关系没有贡献。同样限制在亏格零贡献的作用量 S 就是经典主作用量

$$S_{\text{cl}} = \frac{1}{2}\langle\Psi, Q\Psi\rangle + \sum_{n=3}^{\infty} \frac{1}{n!}\{\Psi^n\}_0. \tag{120}$$

This satisfies the classical master equation (91) on account of the genus zero main identity. Moreover, one can show that this classical master action is invariant under an infinitesimal gauge transformation,

> 借助亏格零主恒等式，它满足经典主方程 (91)。此外，可以证明该经典主作用量在无穷小规范变换下不变,

$$\delta\Psi = Q\Lambda + \sum_{n=1}^{\infty} \frac{1}{n!}[\Lambda, \Psi^n]_0 = Q\Lambda + [\Lambda, \Psi]_0 + \frac{1}{2!}[\Lambda, \Psi, \Psi]_0$$

$$+\frac{1}{3!}[\Lambda, \Psi, \Psi, \Psi]_0 + \cdots, \tag{121}$$

where $|\Lambda\rangle$ is an arbitrary Grassmann odd state in $\mathcal{H}_c$.

> 其中 $|\Lambda\rangle$ 是 $\mathcal{H}_c$ 中任意的格拉斯曼奇态。

String fields of ghost number two are known as classical string fields. Other fields have the interpretation of ghost fields and the anti-fields of the classical string fields and the ghost fields. The classical action is obtained by setting to zero all fields of ghost number $\neq 2$ in the action (120). This action is invariant under a gauge transformation of the form given in (121) with $|\Lambda\rangle$ restricted to be a state of ghost number one. The full BV master action can be regarded as a tool to quantize this classical string field theory.

> 鬼数为 2 的弦场称为经典弦场。其他场分别对应鬼场，以及经典弦场和鬼场的反场。将作用量 (120) 中所有鬼数为 $\neq 2$ 的场置零，即可得到经典作用量。该作用量在 (121) 给出形式的规范变换下不变，其中 $|\Lambda\rangle$ 限定为鬼数为 1 的态。完整 BV 主作用量可视为对该经典弦场论量子化的工具。

Type II Superstring Field Theory

> ## II 型超弦场论

In this section, we follow the approach of [19,20]. In closed superstring field theory, we still use a state space $\mathcal{H}_c$ spanned by states that are annihilated by b_0^- and L_0^- . Moreover, we have to describe four subsectors, the NS-NS, NS-R, R-NS, and R-R sectors of the state space. For each of these subsectors, we define string fields as elements of $\mathcal{H}_c$ with fixed holomorphic and antiholomorphic picture numbers.

在本节中，我们遵循文献 [19,20] 的研究方法。在闭超弦场论中，我们仍使用由被 $\mathcal{H}_c$ 和 b_0^- 湮灭的态张成的态空间 L_0^- 。此外，我们需要描述该态空间的四个子扇区:NS-NS、NS-R、R-NS 和 R-R 扇区。对于每个子扇区，我们将弦场定义为 L_0^- 中具有固定全纯和反全纯鬼数的元素。

As in the case of closed bosonic string field theory, the Grassmanality of a closed superstring field is the Grassmanality of the vertex operator plus that of the target space field. That full Grassmanality is even for the closed superstring field in all four subsectors. This means that the Grassmanality of the vertex operator and that of its target space field are always the same. The usual spin-statistics relation holds, e.g., the classical target space fields (bosons) in the R-R and NS-NS sectors are Grassmann even when the classical target space string fields (fermions) in the R-NS and NS-R sectors are Grassmann odd. This follows because the relevant NS vertex operators are even, while the relevant R vertex operators are odd.

和闭玻色弦场论的情况一样, 闭超弦场的格拉斯曼奇偶性等于顶点算子的格拉斯曼奇偶性加上目标空间场的格拉斯曼奇偶性。四个子扇区的所有闭超弦场的总格拉斯曼奇偶性均为偶。这意味着顶点算子与其对应目标空间场的格拉斯曼奇偶性始终相同。常规自旋统计关系成立: 例如，R-R 和 NS-NS 扇区的经典目标空间场 (玻色子) 是格拉斯曼偶，而 R-NS 和 NS-R 扇区的经典目标空间弦场 (费米子) 是格拉斯曼奇。这是因为相关 NS 顶点算子是格拉斯曼偶，而相关 R 顶点算子是格拉斯曼奇。

The canonical choice is picture number -1 in the NS sector and picture number $-1/2$ or $-3/2$ in the R sector. This is natural because these are the only picture numbers for which the spectrum of the L_0 operator is bounded from below. In other picture numbers, the vacuum fails to be annihilated by one or more β_n or γ_n oscillators with positive n , and by repeatedly applying them on the vacuum, we can get states of arbitrarily negative L_0 eigenvalue.

标准选择是 NS 扇区的鬼数为-1，R 扇区的鬼数为 $-1/2$ 或 $-3/2$ 。这十分自然，因为只有在这些鬼数下，L_0 算符的谱才有下界。在其他鬼数下，真空会被一个或多个带正 β_n 的 γ_n 或 n 振荡子湮灭，反复将这些振荡子作用在真空上，我们就能得到 L_0 本征值任意负的态。

The formulation of string field theory that we are going to describe makes use of the -1 picture states for the NS sector and of both $-1/2$ and -3/2 picture number states in the Ramond sector. Let $\mathcal{H}_{p,q} \subset \mathcal{H}_c$ denote the space of string states of anti-holomorphic picture number p and holomorphic picture number q . Perhaps surprisingly, we introduce two string fields, Ψ and $\widetilde{\Psi}$, each a direct sum over the four sectors of the theory:

我们接下来描述的弦场论表述，在 NS 扇区使用-1 鬼态，在拉蒙德扇区同时使用 $-1/2$ 和-3/2 鬼态。令 $\mathcal{H}_{p,q} \subset \mathcal{H}_c$ 表示反全纯鬼数为 p 、全纯鬼数为 q 的弦态空间。或许出人意料的是，我们引入两个弦场 Ψ 和 $\widetilde{\Psi}$ ，每个都是该理论四个扇区的直和:

$$\text{Type II string fields:} \quad \Psi \in \mathcal{H}_c \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-1/2} \oplus \mathcal{H}_{-1/2,-1} \oplus \mathcal{H}_{-1/2,-1/2}\,, \tag{122}$$

$$\widetilde{\Psi} \in \widetilde{\mathcal{H}}_c \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-3/2} \oplus \mathcal{H}_{-3/2,-1} \oplus \mathcal{H}_{-3/2,-3/2}. \tag{123}$$

Note that Ψ uses the $-1/2$ picture for the R sectors and $\widetilde{\Psi}$ uses the $-3/2$ picture for the R sectors. Since the BPZ inner product between a pair of states requires the picture numbers of the states to add up to $(-2,-2)$, it pairs states in $\mathcal{H}_c$ and $\widetilde{\mathcal{H}}_c$. Both Ψ and $\widetilde{\Psi}$ are Grassmann even.

注意 Ψ 在 R 扇区使用 $-1/2$ 鬼，$\widetilde{\Psi}$ 在 R 扇区使用 $-3/2$ 鬼。由于一对态之间的 BPZ 内积要求两个态的鬼数加和等于 $(-2,-2)$ ，它会匹配 $\mathcal{H}_c$ 和 $\widetilde{\mathcal{H}}_c$ 中的态。Ψ 和 $\widetilde{\Psi}$ 均为格拉斯曼偶。

The construction of the action below only requires multilinear maps to the Grassmann algebra for string fields in $\mathcal{H}_c$. We thus define $\{A_1,\cdots A_n\}$ for $A_i\in\mathcal{H}_c$ in the same way as (107), with $\Omega^{(g,n)}$ given by (58) and $\mathcal{V}_{g,n}$ possibly containing vertical segments. $\mathcal{V}_{g,n}$ is required to satisfy a relation similar to (104), but with a somewhat different definitions of Δ_c and $\{,\}_c$. For this, recall that $\{,\}_c$ denotes the result of gluing two punctures on two Riemann surfaces using the identification $w_1w_2=e^{i\theta}$, and Δ_c denotes the result of gluing two punctures on the same Riemann surface. For superstring theory, there are four different kinds of punctures-they can be of type NSNS, NSR, RNS, and RR, depending on what type of vertex operator is inserted at the puncture. The gluing operations do not mix those different kinds of punctures. If the punctures are of NSNS type, then we continue to use the same definition of $\{,\}_c$ and Δ_c as in the case of bosonic string theory. If the punctures are of RNS, NSR, or RR type, then we insert, respectively, a uniform average of the PCO $\overline{\mathcal{X}},\mathcal{X}$, or both along the gluing circle $|w_1|=1$. As a result, the identity (110) is now replaced by

下文作用量的构造仅要求对 $\mathcal{H}_c$ 中的弦场给出到格拉斯曼代数的多线性映射。因此我们对 $A_i\in\mathcal{H}_c$ 定义 $\{A_1,\cdots A_n\}$ ，方式与 (107) 相同，其中 $\Omega^{(g,n)}$ 由 (58) 给出，且 $\mathcal{V}_{g,n}$ 可包含竖段。$\mathcal{V}_{g,n}$ 需要满足一个和 (104) 类似的关系，但对 Δ_c 和 $\{,\}_c$ 的定义略有不同。在此我们回顾: $\{,\}_c$ 表示利用等同关系 $w_1w_2=e^{i\theta}$ 粘合两个黎曼曲面上两个 puncture 的结果，Δ_c 表示粘合同一个黎曼曲面上两个 puncture 的结果。对超弦理论，共有四种不同类型的 puncture——根据 puncture 处插入的顶点算子类型，可分为 NSNS、NSR、RNS 和 RR 型。粘合操作不会混合这些不同类型的 puncture。若 puncture 为 NSNS 型，我们继续沿用玻色弦理论中对 $\{,\}_c$ 和 Δ_c 的相同定义。若 puncture 为 RNS、NSR 或 RR 型，我们则分别沿粘合圆周 $|w_1|=1$ 插入 PCO $\overline{\mathcal{X}},\mathcal{X}$ 的均匀平均，或同时插入两者。最终，等式 (110) 被替换为

$$
\begin{aligned}
&\sum_{i=1}^{N}\left\{A_1'\ldots A_{i-1}'\left(QA_i'\right)A_{i+1}'\ldots A_N'\right\}\\
&=-\frac{1}{2}\sum_{\substack{\ell,k\geq 0\\ \ell+k=N}}\ \sum_{\substack{\{i_a;a=1,\ldots\ell\}\\ \{j_b:b=1,\ldots k\}}}\left\{A_{i_1}'\ldots A_{i_\ell}'\varphi_s\right\}\left\{\varphi_r A_{j_1}'\ldots A_{j_k}'\right\}\langle\varphi_s^c,\mathcal{G}\varphi_r^c\rangle\\
&\quad-\frac{1}{2}\left\{A_1'\ldots A_N'\varphi_s\varphi_r\right\}\langle\varphi_s^c,\mathcal{G}\varphi_r^c\rangle,
\end{aligned}
\tag{124}
$$

where $|\varphi_s\rangle$ are now basis states in $\mathcal{H}_c$ and $|\varphi_s^c\rangle$ are the basis states in $\widetilde{\mathcal{H}}_c$. The $\mathcal{G}$ operator is defined as follows. We first define the zero modes of the PCOs:

其中 $|\varphi_s\rangle$ 现在是 $\mathcal{H}_c$ 中的基态，$|\varphi_s^c\rangle$ 是 $\widetilde{\mathcal{H}}_c$ 中的基态。$\mathcal{G}$ 算子定义如下: 我们首先定义 PCO 的零模:

$$
\mathcal{X}_0=\oint\frac{dz}{z}\mathcal{X}(z),\ \overline{\mathcal{X}}_0=\oint\frac{d\bar{z}}{\bar{z}}\overline{\mathcal{X}}(\bar{z}).
\tag{125}
$$

Then $\mathcal{G}$ is defined by

随后 $\mathcal{G}$ 由下式定义

$$\mathcal{G} = \begin{cases} \mathbb{1} \text{ on } \mathcal{H}_{-1,-1} \\ \mathcal{X}_0 \text{ on } \mathcal{H}_{-1,-3/2} \\ \overline{\mathcal{X}}_0 \text{ on } \mathcal{H}_{-3/2,-1} \\ \mathcal{X}_0\overline{\mathcal{X}}_0 \text{ on } \mathcal{H}_{-3/2,-3/2} \end{cases} \tag{126}$$

Here1is the identity operator. The effect of $\mathcal{X}_0$ acting on a state inserted at $w = 0$ is precisely to insert a uniform average of a PCO insertion along the circle $|w| = 1$. Note that the operator $\mathcal{G}$ maps $\widetilde{\mathcal{H}}_c$ to $\mathcal{H}_c$ by changing appropriately the picture numbers.

此处 1 是单位算子。$\mathcal{X}_0$ 作用在插入 $w = 0$ 处态上的效果，恰好是沿圆周 $|w| = 1$ 插入一个 PCO 插入的均匀平均。注意算子 $\mathcal{G}$ 通过适当改变鬼数，将 $\widetilde{\mathcal{H}}_c$ 映射到 $\mathcal{H}_c$ 。

In this notation, the BV master action of superstring field theory takes the form: [8]

使用该记号，超弦场论的 BV 主作用量可写为: [8]

$$S = -\frac{1}{2}\langle\widetilde{\Psi}, Q\mathcal{G}\widetilde{\Psi}\rangle + \langle\widetilde{\Psi}, Q\Psi\rangle + \sum_{n=1}^{\infty} \frac{1}{n!}\{\Psi^n\}. \tag{127}$$

Note that $\widetilde{\Psi}$ appears only in the quadratic terms in the action. For this reason, it describes a free field. Therefore, even though we start with double the number of physical string states, half of these states do not take part in the interaction. The other half describes the interacting field theory of type II strings. It is possible to set $\widetilde{\Psi} = \Psi$ in the NSNS sector without changing the interacting part of the theory, but we shall proceed by treating $\widetilde{\Psi}$ and Ψ as independent variables in all four sectors.

注意 $\widetilde{\Psi}$ 仅出现在作用量的二次项中。因此它描述自由场。故而，即便我们初始时拥有两倍数量的物理弦态，这些态中有一半不参与相互作用。另一半描述 II 型弦的相互作用场论。可以在不改变理论相互作用部分的前提下令 NSNS 区的 $\widetilde{\Psi} = \Psi$ 为零，但我们会在全部四个区中将 $\widetilde{\Psi}$ 和 Ψ 视为独立变量进行处理。

The action in (127) satisfies the BV master equation with the definition of ω^{ij} that we give now implicitly as follows. For an arbitrary function $F(\Psi, \widetilde{\Psi})$, let us express the first-order variation of F under arbitrary variation of $\Psi, \widetilde{\Psi}$ as

(127) 的作用量满足 BV 主方程，其中 ω^{ij} 的定义我们现在隐式给出如下。对任意函数 $F(\Psi, \widetilde{\Psi})$ ，我们将 $\Psi, \widetilde{\Psi}$ 任意变化下 F 的一阶变分写为

$$\delta F = \langle F_R, \delta\widetilde{\Psi}\rangle + \langle \widetilde{F}_R, \delta\Psi\rangle = \langle \delta\widetilde{\Psi}, F_L\rangle + \langle \delta\Psi, \widetilde{F}_L\rangle . \tag{128}$$

Then the antibracket of F and G is defined as [19]

则 F 和 G 的反括号定义为 [19]

$$\{F,G\} = -\langle F_R, \widetilde{G}_L\rangle - \langle \widetilde{F}_R, G_L\rangle - \langle \widetilde{F}_R, \mathcal{G}\widetilde{G}_L\rangle . \tag{129}$$

This implicitly defines ω^{rs} via (96), once we write the expansions $\Psi = \sum_r \varphi_r \psi^r$ and $\widetilde{\Psi} = \sum_r \varphi_r^c \widetilde{\psi}^r$, where φ_r and φ_r^c are the basis states satisfying (111). Note however that the use of the basis φ_r for Ψ and φ_r^c for $\widetilde{\Psi}$ is purely a matter of convenience. We could use any other basis for the expansion, and the explicit forms of ω^{rs} and ω_{rs} will be different in another basis.

一旦我们写出展开式 $\Psi = \sum_r \varphi_r \psi^r$ 和 $\widetilde{\Psi} = \sum_r \varphi_r^c \widetilde{\psi}^r$ ，其中 φ_r 和 φ_r^c 是满足式 (111) 的基态，这就通过式 (96) 隐式定义了 ω^{rs} 。但需要注意，为 Ψ 选取基 φ_r 、为 $\widetilde{\Psi}$ 选取基 φ_r^c 纯粹是为了方便。我们可以使用任意其他基做展开，更换基后 ω^{rs} 和 ω_{rs} 的显式形式会发生改变。

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[8] In the NSNS sector, we could set $\widetilde{\Psi} = \Psi$ and work with only one set of fields, but having two sets of fields in all sectors allows us to use a uniform notation.

[8] 在 NSNS sector 中，我们可以设定 $\widetilde{\Psi} = \Psi$ ，只使用一组场来开展工作，但在所有扇区都保留两组场可以让我们使用统一的记号。

We can also generalize (108) to define multilinear products of elements in $\mathcal{H}_c$:

我们还可以推广式 (108)，来定义 $\mathcal{H}_c$ 中元素的多线性乘积:

$$\langle A_1, [A_1, \cdots, A_n]\rangle = \{A_1, \cdots, A_n\} . \tag{130}$$

It follows from picture number conservation rules that $[A_1, \cdots, A_n] \in \widetilde{\mathcal{H}}_c$ for $A_1, \cdots, A_n \in \mathcal{H}_c$. This was expected since the bilinear form couples $\mathcal{H}_c$ to $\widetilde{\mathcal{H}}_c$.

由 picture 数守恒规则可得 $[A_1, \cdots, A_n] \in \widetilde{\mathcal{H}}_c$ 对 $A_1, \cdots, A_n \in \mathcal{H}_c$ 成立。这符合预期，因为双线性型将 $\mathcal{H}_c$ 耦合到 $\widetilde{\mathcal{H}}_c$ 。

As in the case of bosonic string field theory, the classical action is obtained by setting to zero all string fields of ghost number other than two and including contribution from only the genus zero interaction terms. The corresponding action is invariant under the gauge transformation,

和玻色弦场论的情况一样，将鬼数不为 2 的所有弦场都置零，且仅包含零亏格相互作用项的贡献，即可得到经典作用量。对应的作用量在以下规范变换下不变，

$$\delta\,|\widetilde{\Psi}\rangle = Q|\widetilde{\Lambda}\rangle + \sum_{n=1}^{\infty} \frac{1}{n!}[\Lambda\Psi^n]_0,\ \delta\,|\Psi\rangle = Q|\Lambda\rangle + \sum_{n=1}^{\infty} \frac{1}{n!}\mathcal{G}[\Lambda\Psi^n]_0, \tag{131}$$

where $|\Lambda\rangle \in \mathcal{H}_c$, and $|\widetilde{\Lambda}\rangle \in \widetilde{\mathcal{H}}_c$ are arbitrary ghost number one Grassmann odd states. If we relax the constraint on the ghost number but continue to use genus zero interaction terms, we get the classical BV master action satisfying classical BV master equation. The corresponding action has gauge invariance given in (131) with no constraint on the ghost number of Λ .

其中 $|\Lambda\rangle \in \mathcal{H}_c$, and $|\widetilde{\Lambda}\rangle \in \widetilde{\mathcal{H}}_c$ 是任意鬼数为 1 的格拉斯曼奇态。如果放松对鬼数的约束，但仍使用零亏格相互作用项，我们会得到满足经典 BV 主方程的经典 BV 主作用量。对应的作用量具有式 (131) 给出的规范不变性，且对 Λ 的鬼数没有约束。

Heterotic String Field Theory

杂化弦场论

The construction of heterotic string field theory is very similar [19, 20] . Since only the holomorphic sector of the closed string theory has picture number, we have $\mathcal{H}_{-1}$ for NS states and $\mathcal{H}_{-1/2}$ for R states in $\mathcal{H}_c$, and we have $\mathcal{H}_{-1}$ for NS states and $\mathcal{H}_{-3/2}$ for R states in $\widetilde{\mathcal{H}}_c$. As in all closed string field theories, the Grassmanality of the target space field is taken to be the same as that of the vertex operator that it multiplies, resulting in a Grassmann even string field. Again, we use two string fields Ψ and $\widetilde{\Psi}$:

杂化弦场论的构造与 [19, 20] 非常相似。由于只有闭弦理论的全纯部分具有鬼数，我们在 $\mathcal{H}_c$ 中对 NS 态取 $\mathcal{H}_{-1}$ ，对 R 态取 $\mathcal{H}_{-1/2}$ ，在 $\widetilde{\mathcal{H}}_c$ 中对 NS 态取 $\mathcal{H}_{-1}$ ，对 R 态取 $\mathcal{H}_{-3/2}$ 。和所有闭弦场论一样，目标空间场的格拉斯曼性与它所乘的顶点算符一致，因此得到格拉斯曼偶的弦场。同样地，我们使用两个弦场 Ψ 和 $\widetilde{\Psi}$:

$$\text{Heterotic string fields: } \Psi \in \mathcal{H}_c \equiv \mathcal{H}_{-1} \oplus \mathcal{H}_{-1/2}\,, \tag{132}$$

$$\widetilde{\Psi} \in \widetilde{\mathcal{H}}_c \equiv \mathcal{H}_{-1} \oplus \mathcal{H}_{-3/2} \tag{133}$$

The operator $\mathcal{G}$ acting on $\widetilde{\mathcal{H}}$ is defined as identity on $\mathcal{H}_{-1}$ and $\mathcal{X}_0$ on $\mathcal{H}_{-3/2}$:

作用于 $\widetilde{\mathcal{H}}$ 的算符 $\mathcal{G}$ 定义为: 在 $\mathcal{H}_{-1}$ 上是恒等算符，在 $\mathcal{H}_{-3/2}$ 上是 $\mathcal{X}_0$:

$$\mathcal{G} = \begin{cases} 1 \text{ on } \mathcal{H}_{-1}, \\ \mathcal{X}_0 \text{ on } \mathcal{H}_{-3/2}. \end{cases} \tag{134}$$

The action takes the same form as (127):

该作用量与式 (127) 形式相同:

$$S = -\frac{1}{2}\langle \widetilde{\Psi}, Q\mathcal{G}\widetilde{\Psi}\rangle + \langle \widetilde{\Psi}, Q\Psi\rangle + \sum_{n=1}^{\infty} \frac{1}{n!}\{\Psi^n\}. \tag{135}$$

The antibracket also has the same form as (128) and (129). These can be used to write down the antibrackets between the component fields once we use expansions of the string fields analogous to those of the superstring, $\Psi = \sum_r \varphi_r \psi^r$ and $\widetilde{\Psi} = \sum_r \varphi_r^c \widetilde{\psi}^r$

反括号的形式也与式 (128) 和 (129) 相同。当我们采用和超弦类似的弦场展开后，就可以利用它们写出分量场之间的反括号，$\Psi = \sum_r \varphi_r \psi^r$ 和 $\widetilde{\Psi} = \sum_r \varphi_r^c \widetilde{\psi}^r$

Tree-Level Open String Field Theory

树级开弦场论

So far, we have discussed closed string field theory. In the presence of D-branes, string theory also contains open strings whose ends are constrained to move on the D-brane. The dynamics of these open strings is described by open string field theory that we shall now describe. We begin with open bosonic string field theory, and we then briefly note how matters change for open superstring field theory, both at the tree level. Useful references for this section are [5, 9, 21, 56] .

迄今为止，我们已经讨论了闭弦场论。当存在 D 膜时，弦理论还包含开弦，其端点被约束在 D 膜上运动。这些开弦的动力学由开弦场论描述，我们现在就来介绍它。我们首先介绍开玻色弦场论，随后简要说明树级水平下的开超弦场论发生了哪些变化。本节的有用参考文献是 [5, 9, 21, 56] 。

Bosonic Open String Field Theory At the tree level, the procedure for constructing open string field theory is similar to that for closed string field theory with a few differences. A general open string field ψ_o is taken to be an arbitrary open string state without any constraint of type (99), and the inner product $\langle A, B\rangle$ is simply the BPZ inner product $\langle A \mid B\rangle$. We denote the space of open string states by $\mathcal{H}_o$. There is one subtlety with signs: due to the normalization condition (11) or (12), the vacuum expectation value of a product of Grassmann odd operators is Grassmann even. Therefore, we cannot treat both the bra and ket vacuum to be Grassmann even; one of them must be Grassmann odd. We take $|0\rangle$ to be odd and $\langle 0|$ to be even. So if ψ_o denotes the vertex operator representation of the string field, then ψ_o and $|\psi_o\rangle = \psi_o(0)|0\rangle$ have opposite Grassmann parity. This can be encoded in the rule that when a Grassmann odd number passes though), it picks up a minus sign. ψ_o is taken to be a Grassmann odd vertex operator, which means $|\psi_o\rangle$ is a Grassmann even state. In the procedure that we shall follow below to manipulate Grassmann numbers, we shall never need to pass a Grassmann odd number through $\rangle$.

玻色开弦场论在树级，构造开弦场论的流程除少数差异外，与闭弦场论的构造流程类似。一般开弦场 ψ_o 取为任意开弦态，不满足 (99) 类的任何约束，内积 $\langle A, B\rangle$ 就是 BPZ 内积 $\langle A \mid B\rangle$ 。我们用 $\mathcal{H}_o$ 标记开弦态空间。符号问题存在一处微妙之处: 由于归一化条件 (11) 或 (12)，格拉斯曼奇算符乘积的真空期望值是格拉斯曼偶的。因此我们不能同时将左真空和右真空都视作格拉斯曼偶，其中必须有一个是格拉斯曼奇的。我们取 $|0\rangle$ to be odd and $\langle 0|$ 为偶。因此若 ψ_o 表示弦场的顶点算符表示，那么 ψ_o 和 $|\psi_o\rangle = \psi_o(0)|0\rangle$ 的格拉斯曼奇偶性相反。这可以总结为一条规则: 当一个格拉斯曼奇数经过) 时，会获得一个负号。ψ_o 取为格拉斯曼奇顶点算符，这意味着 $|\psi_o\rangle$ 是格拉斯曼偶态。在我们下文处理格拉斯曼数的流程中，永远不需要让格拉斯曼奇数经过 $\rangle$。

The analog of the fiber bundle $\widehat{\mathcal{P}}_{0,n}$ relevant to classical closed string field theory is a fiber bundle $\mathcal{P}_{0,n}^o$ whose base is the moduli space of the upper-half plane with n punctures on the boundary and whose fiber labels the choice of local coordinates at the boundary punctures. We define a $(n-3)$ -form $\Omega_{n-3}^{o(0,n)}$ on $\mathcal{P}_{0,n}^o$ as

经典闭弦场论对应的纤维丛 $\widehat{\mathcal{P}}_{0,n}$ 的类比是纤维丛 $\mathcal{P}^o_{0,n}$，其底流形是边界上带有 n 个 puncture 的上半平面的模空间，纤维标记边界 puncture 处局部坐标的选择。我们在 $\mathcal{P}^o_{0,n}$ 上定义一个 $(n-3)$ 形式 $\Omega^{o(0,n)}_{n-3}$ 为

$$\Omega^{o(0,n)}_{n-3}(A_1,\cdots,A_n) \equiv K^{-1}\widehat{\Omega}^{(0,1,0,n)}_{n-3}(A_1,\cdots,A_n), \tag{136}$$

where $\widehat{\Omega}^{(0,1,0,n)}_{n-3}$ is the form defined in (68) following the sign convention described in section ”Signs of Forms in $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ ”, and the explicit factor of K^{-1} implies that the correlators are effectively computed using the normalization (11). Also we have dropped the normalization constant $N_{0,1,0,n}$. These differences can be traced to the use of an open string coupling constant g_o introduced below instead of $g_s^{1/2}$ used in (71). More discussion on this can be found in section ”Results for Special Amplitudes”.

其中 $\widehat{\Omega}^{(0,1,0,n)}_{n-3}$ 是遵循“$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 中形式的符号”一节所述符号约定、由 (68) 定义的形式，K^{-1} 的显式因子意味着关联函数实际上是用归一化 (11) 计算的。同时我们省略了归一化常数 $N_{0,1,0,n}$ 。这些差异源于我们下文引入开弦耦合常数 g_o ，而非 (71) 中使用的 $g_s^{1/2}$ ，更多相关讨论可以参见“特殊振幅的结果”一节。

Next, in analogy with (104), we introduce $(n-3)$ -dimensional chains $\mathcal{V}^o_{0,n}$ of $\mathcal{P}^o_{0,n}$, satisfying

接下来，类似 (104)，我们引入 $\mathcal{P}^o_{0,n}$ 的 $(n-3)$ 维链 $\mathcal{V}^o_{0,n}$ ，满足

$$\partial\mathcal{V}^o_{0,n} = -\frac{1}{2}\sum_{\substack{n_1,n_2\\ n_1+n_2=n+2}} \left\{\mathcal{V}^o_{0,n_1},\mathcal{V}^o_{0,n_2}\right\}_o. \tag{137}$$

The definition of $\left\{\mathcal{V}^o_{0,n_1},\mathcal{V}^o_{0,n_2}\right\}_o$ is similar to the one given below (104), except that the punctures that are glued are boundary punctures, and (106) is replaced by $w_1w_2=-1$. We now define $\{A_1,\cdots,A_n\}$ as in (107):

$\left\{\mathcal{V}^o_{0,n_1},\mathcal{V}^o_{0,n_2}\right\}_o$ 的定义与下文 (104) 给出的定义类似，区别在于被粘合的 puncture 是边界 puncture，且 (106) 被替换为 $w_1w_2=-1$ 。现在我们按照 (107) 定义 $\{A_1,\cdots,A_n\}$:

$$\{A_1,\cdots,A_n\} \equiv g_o^{n-2}\int_{\mathcal{V}^o_{0,n}} \Omega^{o(0,n)}_{n-3}(A_1,\cdots,A_n), \tag{138}$$

which uses an open string coupling g_o , and write the open string field theory action as

其中用到开弦耦合常数 g_o ，并将开弦场论作用量写为

$$S_o = \frac{1}{2}\langle\psi_o, Q\psi_o\rangle' + \sum_{n=3}^{\infty}\frac{1}{n!}\{\psi_o^n\}. \tag{139}$$

Q now denotes the BRST charge in the open string theory, and g_o is the open string coupling constant. Its relation to the closed string coupling constant g_s depends on the specific D-brane we consider, but a general relation was derived in [57]:

Q 此处表示开弦理论中的 BRST 荷，g_o 是开弦耦合常数。它与闭弦耦合常数 g_s 的关系依赖于我们考虑的具体 D 膜，但文献 [57] 已经推导出了一个通用关系:

$$\mathcal{T} = \frac{1}{2\pi^2 g_o^2} \tag{140}$$

where $\mathcal{T}$ is the tension of the D-brane. We shall give a different derivation of this result in section "Results for Special Amplitudes".

其中 $\mathcal{T}$ 是 D 膜的张力。我们会在“特殊振幅的结果”一节给出该结果的另一种推导。

As discussed at the end of section "Bosonic String Amplitudes and Their Off-Shell Generalization", the chain $\mathcal{V}^o_{0,n}$ appearing in (138) has disconnected components that differ from each other by different cyclic ordering of the vertex operators $A_1, \cdots, A_n$ on the boundary. We have to add the contribution from these disconnected components with appropriate relative signs such that for Grassmann odd open string vertex operators A_i ’s, $\{A_1 \cdots A_N\}$ is symmetric under arbitrary permutation of the A_i ’s. If it is anti-symmetric under the exchange of any pair of A_i ’s, the interaction terms in the action (139) will vanish identically. The rules for $A_i \leftrightarrow A_j$ for other choice of statistics can be found by multiplying the even A_i ’s by Grassmann odd variables and then applying the rules for odd A_i . This gives

正如“玻色弦振幅及其离壳推广”一节末尾所讨论的，(138) 中出现的链 $\mathcal{V}^o_{0,n}$ 存在不连通分支，这些分支的区别在于顶点算符 $A_1, \cdots, A_n$ 在边界上的循环排序不同。我们必须为这些不连通分支加上带有合适相对符号的贡献，才能保证对于格拉斯曼奇的开弦顶点算符 A_i ，$\{A_1 \cdots A_N\}$ 在 A_i 的任意置换下保持对称。如果 $\{A_1 \cdots A_N\}$ 在任意一对 A_i 交换下反对称，那么作用量 (139) 中的相互作用项就会恒等于零。其他统计性质选择下 $A_i \leftrightarrow A_j$ 的规则可以通过如下方式得到: 给偶格拉斯曼性的 A_i 乘上格拉斯曼奇变量，再应用奇 A_i 对应的规则，最终得到

$$\{A_1 \cdots A_i A_{i+1} \cdots A_n\} = (-1)^{A_i A_{i+1}+1} \{A_1 \cdots A_{i+1} A_i \cdots A_n\}. \tag{141}$$

We can also write down the analog of the closed string main identity (110) as follows. Given $A_1, \cdots, A_N \in \mathcal{H}_o$ carrying arbitrary Grassmann parity, we first define $A_i' = \varepsilon_i A_i$ where ε_i ’s are chosen so that each A_i' is Grassmann odd. The main identity now takes the form

我们也可以写下闭弦主恒等式 (110) 的开弦类比形式如下。给定携带任意格拉斯曼奇偶性的 $A_1, \cdots, A_N \in \mathcal{H}_o$ ，我们先定义 $A_i' = \varepsilon_i A_i$ ，其中对 ε_i 的选择要保证每个 A_i' 都是格拉斯曼奇的。此时主恒等式的形式为

$$\begin{aligned} Q\left[A_1' \cdots A_N'\right] &= \sum_{i=1}^{N} (-1)^{i-1} \left[A_1' \ldots A_{i-1}' \left(QA_i'\right) A_{i+1}' \ldots A_N'\right] \\ &+ \sum_{\substack{\ell,k\geq 0 \\ \ell+k=N}} \sum_{\substack{\{i_a;a=1,\ldots\ell\} \\ \{j_b:b=1,\ldots k\} \\ \{i_a\}\cup\{j_b\}=\{1,\ldots N\}}} \left[\left[A_{j_1}' \ldots A_{j_k}'\right] A_{i_1}' \ldots A_{i_\ell}'\right], \end{aligned} \tag{142}$$

with products $[A_1, \cdots, A_n]$ defined analogously to (108):

其中乘积 $[A_1, \cdots, A_n]$ 的定义与 (108) 类似:

$$\langle A_0, [A_1, \cdots, A_n]\rangle' = \{A_0, A_1, \cdots, A_n\}. \tag{143}$$

These products, just as the multilinear functions $\{\cdots\}$ defined above, include sums over different cyclic orderings.

这些乘积和上文定义的多线性函数 $\{\cdots\}$ 一样，包含对不同循环排序的求和。

Unlike in the case of closed strings (see Remark 5, section "Closed Bosonic String Field Theory"), for open strings, [is Grassmann even. This may appear counterintuitive since from (68), it would seem that a Grassmann odd number multiplying the A_i ' s have to pass through the $\mathcal{B}$ insertions before it can come out to the left, picking additional signs. However, we recall the sign prescription for $\widehat{\Omega}$ given at the end of section "Signs of Forms in $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ ", namely, that we start with operators carrying the "correct" Grassmann parity; arrange the operators inside the correlation function so that all the vertex operators appear on the extreme left followed by the $\mathcal{B}$ insertions, picking up any signs that may appear from the rearrangement; and then declare that the same form of $\widehat{\Omega}$ is valid even when the operators have "wrong" Grassmann parity. Since the vertex operators appear on the extreme left, it follows that no extra sign appears while passing a Grassmann odd element to the left, other than those from having to pass it through some of the A_i ' s. The same logic tells us that we can move a Grassmann odd operator through {without picking up any extra sign. The other useful information is that if $A_1, \cdots, A_n$ are Grassmann odd, then $[A_1, \cdots, A_n]$ is Grassmann even. With this knowledge, we can find the identity (142) for A_i ' s of arbitrary Grassmann parity by moving the ε_i ' s to the left and picking up the signs we encounter on the way.

与闭弦的情况不同 (参见“闭玻色弦场论”一节的注 5)，对开弦而言，[是格拉斯曼偶的。这一点可能看起来有违直觉，因为从式 (68) 看，一个格拉斯曼奇数乘以 A_i ，似乎必须穿过 $\mathcal{B}$ 插入才能到达左侧，会带来额外的符号。但我们记得“$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 中形式的符号”一节末尾给出的 $\widehat{\Omega}$ 符号规定: 即我们从携带“正确”格拉斯曼奇偶性的算符出发；将关联函数内的算符排序，让所有顶点算符都在最左侧，后跟 $\mathcal{B}$ 插入，整理过程中收集所有产生的符号；然后声明即使算符具有“错误的”格拉斯曼奇偶性，$\widehat{\Omega}$ 的同样形式依然成立。由于顶点算符位于最左侧，因此将格拉斯曼奇元素移到左侧时，除了穿过部分 A_i 产生的符号外，不会产生额外符号。同样的逻辑告诉我们，移动格拉斯曼奇算符穿过 {时不会拾取任何额外符号。另一项有用的结论是: 若 $A_1, \cdots, A_n$ 是格拉斯曼奇的，则 $[A_1, \cdots, A_n]$ 是格拉斯曼偶的。有了这些结论，我们通过将 ε_i 移到左侧并收集途中得到的符号，就能推导出任意格拉斯曼奇偶性的 A_i 满足的恒等式 (142)。

The action (139) satisfies the classical BV master equation under the following definition of the antibracket. If for an arbitrary function $F(\Psi^o)$, one expresses the first-order variation as

作用量 (139) 在下述反括号的定义下满足经典 BV 主方程。若对任意函数 $F(\Psi^o)$ ，可将一阶变分写为

$$\delta F = \left\langle F_R^o, \delta\psi_o\right\rangle' = \left\langle \delta\psi_o, F_L^o\right\rangle', \tag{144}$$

then the antibracket of F and G is defined as

则 F 和 G 的反括号定义为

$$\{F, G\} = -\langle F_R^o, G_L^o \rangle'. \tag{145}$$

At the level of target space fields, the above expressions lead to concrete expressions for the antibracket when using the string field expansion $\psi_o = \sum_r \hat{\varphi}_r \psi^r$, with the $\hat{\varphi}_r$ forming a complete basis of states of the BCFT.

在目标空间场层面，当使用弦场展开 $\psi_o = \sum_r \hat{\varphi}_r \psi^r$ 时，上述表达式给出反括号的具体形式，其中 $\hat{\varphi}_r$ 构成 BCFT 态的一组完备基。

The classical master action also has a gauge invariance analogous to (121). The infinitesimal gauge transformations leaving the action invariant use a Grassmann even string field gauge-parameter Λ . The gauge transformation takes the form

经典主作用量也具有类似 (121) 的规范不变性。保持作用量不变的无穷小规范变换使用格拉斯曼偶的弦场规范参数 Λ ，规范变换形式为

$$\delta\Phi = Q\Lambda + [\Lambda\Phi] + \frac{1}{2!}[\Lambda, \Phi, \Phi] + \cdots. \tag{146}$$

Classical string field corresponds to an open string state of ghost number one, and the corresponding gauge transformation laws are generated by open string states of ghost number zero.

经典弦场对应鬼数为 1 的开弦态，对应的规范变换规律由鬼数为 0 的开弦态生成。

The action (139) does not satisfy the quantum BV master equation (92) since the term involving the Δ operation is missing in the geometric equation (137). There is one exception to this, provided by Witten's cubic open string field theory. This can be regarded as a special case of the theory described above, where

作用量 (139) 不满足量子 BV 主方程 (92)，因为几何方程 (139) 中缺失了涉及 Δ 运算的项。但威滕的三次开弦场论是唯一例外，它可以看作上文描述理论的一个特例，其中

$$[A_1, A_2] = A_1 \star A_2 - (-1)^{A_1 A_2} A_2 \star A_1. \tag{147}$$

Here, the star product $A_1 \star A_2$ is a particular map from the tensor product of two open string state spaces to the open string state space and A_i in the exponent is zero (one) if A_i carries even (odd) ghost number. The difference between $[A_2, A_3]$ and $A_2 \star A_3$ is that when we take the inner product with another state A_1 , the former gives the full contribution to the disk three-point function of A_1, A_2 , and A_3 with some particular choice of local coordinates at the punctures, while the latter gives the contribution to the disk three-point function of A_1, A_2 , and A_3 for a particular cyclic ordering. It can be shown that we can define $A_1 \star A_2$ with appropriate choice of local coordinates on the three-string vertex $\mathcal{V}_{0,3}^o$ so that [5]

此处，星乘积 $A_1 \star A_2$ 是从两个开弦态空间的张量积到开弦态空间的特定映射，且若 A_i 携带偶 (奇) 鬼数，指数中的 A_i 为零 (一)。$[A_2, A_3]$ 和 $A_2 \star A_3$ 的区别在于: 当我们取和另一个态 A_1 的内积时，前者对带特殊选择穿刺处局部坐标的 A_1, A_2 、A_3 的圆盘三点函数给出全贡献，而后者对特定循环序的 A_1, A_2 、A_3 的圆盘三点函数给出贡献。可以证明，我们可以通过对三弦顶点 $\mathcal{V}_{0,3}^o$ 选择合适的局部坐标来定义 $A_1 \star A_2$ ，使得 [5]

$$A_1 \star (A_2 \star A_3) = (A_1 \star A_2) \star A_3. \tag{148}$$

As will become clear in section "String Vertices", geometrically, this translates to the statement that $\{\mathcal{V}_{0,3}^o, \mathcal{V}_{0,3}^o\}_o$ vanishes, and we can solve (137) by setting $\mathcal{V}_{0,n}^o$ to zero for $n \geq 4$. Therefore, we have

正如我们将在“弦顶点”一节中说明的，从几何上看，这对应于 $\{\mathcal{V}_{0,3}^o, \mathcal{V}_{0,3}^o\}_o$ 为零的结论，我们可以通过令 $\mathcal{V}_{0,n}^o$ 为零来求解 $n \geq 4$ 的情况。因此我们得到

$$[A_1, A_2, \cdots, A_n] = 0 \text{ for } n \geq 3, \tag{149}$$

and the action terminates at the cubic order. As we discuss below, formally ΔS_o also vanishes, and the action satisfies the full quantum BV master equation, but this requires dropping boundary contributions from degenerate Riemann surfaces.

且作用量在三次阶截断。如下文所述，形式上 ΔS_o 也为零，作用量满足完整量子 BV 主方程，但这需要去掉退化黎曼曲面的边界贡献。

In the notation explained at the end of section "World-Sheet Conventions", the star product can be defined via the relation

采用“世界面规范约定”一节末尾说明的记号，星积可以通过如下关系定义

$$\langle A_1, A_2 \star A_3 \rangle' = \langle f_1 \circ A_1(0)\, f_2 \circ A_2(0)\, f_3 \circ A_3(0) \rangle'_{\mathrm{UHP}}\,, \tag{150}$$

where the subscript UHP indicates that the correlation function is computed on the upper-half plane and the maps f_i can be written as follows:

其中下标 UHP 表示关联函数在上半平面计算，映射 f_i 可以写为:

$$\begin{gathered} f_1(w_1) = h^{-1}\left(e^{2\pi i/3}(h(w_1))^{2/3}\right) \\ f_2(w_2) = h^{-1}\left((h(w_2))^{2/3}\right),\ h(u) \equiv \frac{1+iu}{1-iu}, \\ f_3(w_3) = h^{-1}\left(e^{-2\pi i/3}(h(w_3))^{2/3}\right). \end{gathered} \tag{151}$$

More discussion on these maps can be found around (472).

关于这些映射的更多讨论可以在 (472) 附近找到。

An interesting application of the cubic open string field theory for computing the energy of the rolling tachyon solution can be found in [58]. Other applications will be discussed in section "Applications of String Field Theory".

三次开弦场论有一个有趣的应用，即计算滚动快子解的能量，可见文献 [58]。其他应用将在“弦场论的应用”一节讨论。

Open Superstring Field Theory The construction of tree-level open superstring field theory, describing open string dynamics on a D-brane of superstring theory, is similar. As we did in closed string theory, we introduce two sets of string fields $\psi_o \in \mathcal{H}_o$ and $\widetilde{\psi}_o \in \widetilde{\mathcal{H}}_o$, each with an NS and R sector. Here $\mathcal{H}_o$ contains open string states of picture number -1 or $-1/2$ and $\widetilde{\mathcal{H}}_o$ contains open string states of picture number -1 or -3/2:

开超弦场论描述超弦理论 D 膜上开弦动力学的树级开超弦场论构造是类似的。和我们处理闭弦理论时一样，我们引入两组弦场 $\psi_o \in \mathcal{H}_o$ 和 $\widetilde{\psi}_o \in \widetilde{\mathcal{H}}_o$ ，每组都包含 NS sector 和 R sector。这里 $\mathcal{H}_o$ 包含鬼数为-1 或 $-1/2$ 的开弦态，$\widetilde{\mathcal{H}}_o$ 包含鬼数为-1 或-3/2 的开弦态:

Open superstring classical fields: $\psi_o \in \mathcal{H}_o \equiv \mathcal{H}_{-1} \oplus \mathcal{H}_{-1/2}$,

开超弦经典场: $\psi_o \in \mathcal{H}_o \equiv \mathcal{H}_{-1} \oplus \mathcal{H}_{-1/2}$,

(152)

$$\widetilde{\psi}_o \in \widetilde{\mathcal{H}}_o \equiv \mathcal{H}_{-1} \oplus \mathcal{H}_{-3/2} \tag{153}$$

The construction of $\{A_1, \cdots, A_n\}$ follows a procedure that combines the features of open bosonic string field theory and closed superstring field theory, with insertion of picture changing operators, and the tree-level open superstring field theory action is given by,

$\{A_1, \cdots, A_n\}$ 的构造结合了开玻色弦场论和闭超弦场论的特点，需要插入图片变换算子，树级开超弦场论的作用量为

$$S = -\frac{1}{2}\langle\widetilde{\psi}_o, Q\mathcal{G}\widetilde{\psi}_o\rangle' + \langle\widetilde{\psi}_o, Q\psi_o\rangle' + \sum_{n=1}^{\infty}\frac{1}{n!}\{\psi_o^n\}. \tag{154}$$

As usual, $\mathcal{G}$ is defined as the identity operator on the NS sector states and as the zero mode $\mathcal{X}_0$ of the PCO on the R sector states.

和通常一样，$\mathcal{G}$ 定义为 NS sector 态上的恒等算子，在 R sector 态上定义为 PCO 的零模 $\mathcal{X}_0$ 。

Unlike bosonic open string field theory, there is no known version of classical open superstring field theory where the action terminates at the cubic order and satisfies the classical BV master equation. Higher-order interactions are needed for this. As discussed below, the full open-closed string field theory, with its higher genus contributions, is needed to satisfy the quantum BV master equation.

和玻色开弦场论不同，目前还没有已知的经典开超弦场论版本能让作用量在三次阶截断同时满足经典 BV 主方程，为此需要引入更高阶相互作用。如下文所述，要满足量子 BV 主方程，需要包含高亏格贡献的完整开-闭弦场论。

If we are working in the NS sector and are willing to use the full freedom of choosing the local coordinates at the punctures and possibly use vertical integration, there are many non-canonical choices of PCO insertions that will lead to the construction of $\{A_1 \cdots A_n\}$ and $[A_1 \cdots A_n]$ satisfying (142),(143). The construction of [59, 60] , giving a canonical choice of PCO locations, can be regarded as a simplification of this more general class of theories. This will be reviewed in section "String Vertices for Open Superstring Field Theory". For the R sector, however, we need the doubling of fields to write an action.

如果我们在 NS sector 中工作，并且愿意充分利用选择穿刺点局部坐标的自由度，还可以使用竖直积分，那么存在许多非规范的 PCO 插入选择，都可以构造出满足 (142)、(143) 的 $\{A_1 \cdots A_n\}$ 和 $[A_1 \cdots A_n]$ 。给出 PCO 位置规范选择的 [59, 60] 的构造，可以看作是这类更一般理论的简化。我们会在“开超弦场论的弦顶点”一节回顾这一点。但对于 R sector，我们需要加倍场量才能写出作用量。

Having discussed quantum closed string field theory and classical open string field theory, we can ask: What is the quantum version of the classical open string field theory? Since, as claimed, this classical action S satisfies $\{S, S\} = 0$, we ask: Does S satisfy $\Delta S = 0$ as well, making it suitable for a quantum theory? The answer, in general, is no, it does not. This is clear geometrically in the versions of classical open-string field theory with higher products.

讨论完量子闭弦场论和经典开弦场论，我们可以提出问题: 经典开弦场论的量子版本是什么？如前所述，这个经典作用量 S 满足 $\{S, S\} = 0$ ，那么我们要问: S 是否也满足 $\Delta S = 0$ ，从而适合作为量子理论？一般来说，答案是否定的。在含高阶乘积的经典开弦场论版本中，这一点从几何上看很清楚。

When one considers the associative bosonic open string field theory of Witten, the answer is less clear. When one computes open string amplitudes using this theory, no additional vertices are needed to cover the moduli spaces of higher genus surfaces with one or more boundaries. This would suggest that for the associative vertex $\mathcal{V}^o_{0,3}$, one has $\Delta\mathcal{V}^o_{0,3} = 0$, formally. This claim is somewhat surprising since $\Delta\mathcal{V}^o_{0,3}$ is a degenerate surface, being the boundary of the (real) one-dimensional moduli space of an annulus with one open-string puncture. The degenerate surface is one with two disks touching at an interior node, the first disk with one boundary puncture and the other disk with no puncture. In fact, it has been argued that ΔS is actually singular [61, 62] .

在研究威滕的结合型玻色开弦场论时，答案并不清晰。用该理论计算开弦振幅时，不需要额外顶点就能覆盖带有一个或多个边界的亏格更高曲面的模空间。这说明对于结合顶点 $\mathcal{V}^o_{0,3}$ ，形式上存在 $\Delta\mathcal{V}^o_{0,3} = 0$ 。这个结论相当出人意料，因为 $\Delta\mathcal{V}^o_{0,3}$ 是一个退化曲面，它是带有一个开弦 puncture 的环带的 (实) 一维模空间的边界。该退化曲面是两个圆盘在内部结点相接，其中一个圆盘带有一个边界 puncture，另一个圆盘没有 puncture。事实上已有研究指出 ΔS 实际上是奇异的 [61, 62] 。

The question is how to supplement classical open string field theory in order to get a theory that manifestly solves the master equation. The physical intuition points the way. Open strings can close by joining their endpoints and becoming closed strings. Therefore, the states of closed strings must be added to the open string theory. The closed string states will have self-interactions-those of closed string field theory—as well as interactions with open strings. Open strings will also have higher-order interactions among themselves. All these interactions belong to the quantum theory. This is the quantum open-closed string field theory that we shall discuss next.

问题在于如何补充经典开弦场论，以得到一个能明显满足主方程的理论。物理直觉指明了方向: 开弦的端点可以接合，闭合成为闭弦，因此必须在开弦理论中加入闭弦态。闭弦态不仅存在自相互作用——也就是闭弦场论中的自相互作用——还会和开弦发生相互作用。开弦自身也存在高阶相互作用。所有这些相互作用都属于量子理论，这就是我们接下来要讨论的量子开-闭弦场论。

Open-Closed String Field Theory

开弦-闭弦场论

As the name suggests, open-closed string field theory is the field theory of closed and open strings. This describes the full quantum theory in the presence of D-branes. The construction of the theory combines the ingredients already described in the previous sections. We shall describe the construction in superstring field theory; the result for the bosonic string theory can be obtained by setting $\widetilde{\Psi}$ equal to Ψ in the final formula. Useful references for this section are [7-9,21], the first three of which are in the context of bosonic string theory.

顾名思义，开弦-闭弦场论是同时描述开弦与闭弦的场论，它给出了存在 D 膜时的完整量子理论。该理论的构造结合了前面章节已经介绍过的要素。我们下文将围绕超弦场论的构造展开论述；玻色弦理论的结果可以通过在最终公式中令 $\widetilde{\Psi}$ 等于 Ψ 得到。本节的有用参考文献为 [7-9,21]，其中前三篇是在玻色弦理论的框架下讨论的。

In the closed string sector, we have two sets of string fields, Ψ_c and $\widetilde{\Psi}_c$, and we also have two set of string fields, Ψ_o and $\widetilde{\Psi}_o$, in the open string sector:

在闭弦 sector，我们有两组弦场 Ψ_c 和 $\widetilde{\Psi}_c$ ；在开弦 sector，我们也有两组弦场 Ψ_o 和 $\widetilde{\Psi}_o$:

$$\text{Open-closed string fields: } \Psi_c \in \mathcal{H}_c,\ \widetilde{\Psi}_c \in \widetilde{\mathcal{H}}_c\,,$$

$$\Psi_o \in \mathcal{H}_o,\ \widetilde{\Psi}_o \in \widetilde{\mathcal{H}}_o \tag{155}$$

Here, $(\mathcal{H}_c, \widetilde{\mathcal{H}}_c)$ are those as in type II in (122) and (123), and $(\mathcal{H}_o, \widetilde{\mathcal{H}}_o)$ are those as in classical open superstring theory in (152) and (153). The relevant surfaces have boundaries and carry closed and open string punctures. As defined before, $\mathcal{M}_{g,b,n_c,n_o}$ is the moduli spaces of Riemann surfaces of genus g and b boundaries, with n_c closed string punctures and n_o open string punctures. It has real dimension d_{g,b,n_c,n_o} , given in (61).

此处 $(\mathcal{H}_c, \widetilde{\mathcal{H}}_c)$ 对应式 (122) 和 (123) 中 II 型的情形，$(\mathcal{H}_o, \widetilde{\mathcal{H}}_o)$ 对应式 (152) 和 (153) 中经典开超弦理论的情形。相关曲面带有边界，同时包含闭弦孔和开弦孔。按照前文的定义，$\mathcal{M}_{g,b,n_c,n_o}$ 是亏格为 g 、带有 b 个边界、n_c 个闭弦孔和 n_o 个开弦孔的黎曼曲面的模空间，其实维数为 d_{g,b,n_c,n_o} ，已由式 (61) 给出。

We define $\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$ to be a fiber bundle whose base is $\mathcal{M}_{g,b,n_c,n_o}$ and whose fiber contains information on the local coordinate system at the punctures and the PCO locations. The relevant p -form $\Omega_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots A_{n_c}^c;A_1^o,\cdots .A_{n_o}^o\right)$ on $\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$ is given in (79). We also construct d_{g,b,n_c,n_o} -dimensional chains $\mathcal{V}_{g,b,n_c,n_o}$ in $\widetilde{\mathcal{P}}^s_{g,b,n_c,n_o}$, satisfying

我们将 $\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$ 定义为一个纤维丛，其底空间为 $\mathcal{M}_{g,b,n_c,n_o}$ ，纤维包含孔处局部坐标系和 PCO 位置的信息。$\widehat{\mathcal{P}}^s_{g,b,n_c,n_o}$ 上相关的 p -形式 $\Omega_p^{(g,b,n_c,n_o)}\left(A_1^c,\cdots A_{n_c}^c;A_1^o,\cdots .A_{n_o}^o\right)$ 由式 (79) 给出。我们还在 $\widetilde{\mathcal{P}}^s_{g,b,n_c,n_o}$ 中构造了 d_{g,b,n_c,n_o} 维链 $\mathcal{V}_{g,b,n_c,n_o}$ ，满足

$$\partial\mathcal{V}_{g,b,n_c,n_o}=-\Delta_c\mathcal{V}_{g-1,b,n_c+2,n_o}-\Delta_o'\mathcal{V}_{g,b-1,n_c,n_o+2}-\Delta_o\mathcal{V}_{g-1,b+1,n_c,n_o+2}$$

$$-\frac{1}{2}\sum_{\substack{g_1+g_2=g,\,b_1+b_2=b\\ n_{c1}+n_{c2}=n_c+2,\,n_{o1}+n_{o2}=n_o}}\left\{\mathcal{V}_{g_1,b_1,n_{c1},n_{o1}},\mathcal{V}_{g_2,b_2,n_{c2},n_{o2}}\right\}_c$$

$$-\frac{1}{2}\sum_{\substack{g_1+g_2=g,\,b_1+b_2=b+1\\ n_{c1}+n_{c2}=n_c,\,n_{o1}+n_{o2}=n_o+2}}\left\{\mathcal{V}_{g_1,b_1,n_{c1},n_{o1}},\mathcal{V}_{g_2,b_2,n_{c2},n_{o2}}\right\}_o. \tag{156}$$

Here Δ_c and $\{,\}_c$, with c for closed, denote the usual twist gluing of closed string punctures described below (104) while Δ_o',Δ_o and $\{,\}_o$ denote gluing of open string punctures via $w_1w_2=-1.\Delta_o'$ glues two open string punctures on the same boundary, increasing the number of boundary components by one, whereas Δ_o glues two open string punctures lying on different boundaries, decreasing the number of boundary components by one and increasing the genus by one (the pictorial representation of the above identity is given in Fig. 5, section "Geometric BV Master Equation and String Field Theory Master Equation"). We now define the multilinear functions

此处 Δ_c 和 $\{,\}_c$ (闭弦对应 c) 表示式 (104) 下方介绍的闭弦孔常规扭转粘合，而 Δ_o',Δ_o 和 $\{,\}_o$ 表示开弦孔的粘合: 其中 $w_1w_2=-1.\Delta_o'$ 粘合同一边界上的两个开弦孔，边界分支数增加 1；而 Δ_o 粘合不同边界上的两个开弦孔，边界分支数减少 1，亏格增加 1(上述恒等式的图示参见“几何 BV 主方程与弦场论主方程”一节的图 5)。我们现在定义多重线性函数

$$\left\{A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right\}=\sum_{g=0}^{\infty}\sum_{b=0}^{\infty}\left\{A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right\}_{g,b},$$

$$\left\{A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right\}_{g,b}$$

$$=(g_s)^{-\chi_{g,b,n_c,n_o}}\int_{\mathcal{V}_{g,b,n_c,n_o}}\Omega^{(g,b,n_c,n_o)}_{d_{g,b,n_c,n_o}}\left(A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\right).$$

(157)

Note that in defining the above multilinear functions, we are using the closed string coupling g_s . The open string coupling g_o we used to write classical open string field does not appear here, even for the purely open string couplings on a disk. This is possible because $g_o\sim\sqrt{g_s}$. At the end of this subsection, we shall discuss how to recover the classical open string field theory action of section "Tree-Level Open String Field Theory" from the open closed string field theory action discussed here. There we shall also determine the relation between g_s and g_o .

请注意，在定义上述多重线性函数时，我们使用了闭弦耦合常数 g_s 。我们用来写出经典开弦场的开弦耦合常数 g_o 并未出现在此处——即使对于圆盘上的纯开弦耦合也是如此。这之所以成立，是因为 $g_o \sim \sqrt{g_s}$ 。在本小节末尾，我们将讨论如何从此处讨论的开-闭弦场论作用量还原得到“树级开弦场论”一节的经典开弦场论作用量。我们还会在那里确定 g_s 和 g_o 之间的关系。

There is one more quantity that requires special attention-the disk one-point function of a closed string. Before dealing with the details of this, let us recall from section "Signs of Forms in $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ " that the boundary state $|B\rangle$ is defined such that $\langle B \mid \chi\rangle$ gives the disk one-point function of a closed string state χ . If we take a closed string state $|\phi\rangle$ satisfying (99) and consider the one-point function of $c_0^-|\phi\rangle$ on the unit disk as above, then the disk one-point function is given by $\langle B|c_0^-|\phi\rangle = \langle B,\phi\rangle = \langle\phi,B\rangle$.

还有一个量需要特别注意——闭弦的圆盘单点函数。在讨论具体细节之前，我们先回顾“$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 中形式的符号”一节的内容: 边界态 $|B\rangle$ 的定义满足 $\langle B \mid \chi\rangle$ 给出闭弦态 χ 的圆盘单点函数。如果我们如上述取单位圆盘上的闭弦态 $|\phi\rangle$ satisfying (99) and consider the one-point function of $c_0^-|\phi\rangle$ ，则圆盘单点函数由 $\langle B|c_0^-|\phi\rangle = \langle B,\phi\rangle = \langle\phi,B\rangle$. 给出。

The disk one-point function is special for two reasons. First of all, of all the interaction vertices of open-closed string field theory, this is the only one that has a conformal Killing vector. If the closed string vertex operator is taken to be at the center of the disk, then this is just the rotation about the center. Second, if we take the closed string state to be in $\mathcal{H}_c$ so that the RR sector states have picture number $(-1/2,-1/2)$, then we get a total picture number -1 from the vertex operator after combining the contribution from the left and the right sector. Since on the disk a nonvanishing amplitude must have total picture number -2, we will need to insert an inverse picture changing operator carrying picture number -1 to get a non-vanishing result. However, the zero mode of the inverse picture changing operator does not have nice properties—in particular, it does not commute with b_0^- . For this reason, we use the vertex operators corresponding to the string field $\widetilde{\Psi} \in \widetilde{\mathcal{H}}_c$ truncated to the NS-NS and RR sectors $\left(\mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-3/2,-3/2}\right)$ to construct the disk one-point function and define

圆盘单点函数的特殊性源于两点: 首先，在开-闭弦场论的所有相互作用顶点中，这是唯一一个共形 Killing 矢量的顶点。如果将闭弦顶点算子放在圆盘中心，那么这个共形 Killing 矢量就是绕中心的旋转。其次，如果我们取闭弦态处于 $\mathcal{H}_c$ 中，此时 RR 扇区态的鬼数为 $(-1/2,-1/2)$ ，那么合并左右扇区的贡献后，我们从顶点算子得到总鬼数为-1。由于圆盘上非零振幅的总鬼数必须为-2，我们需要插入一个携带-1 鬼数的逆图变算子来得到非零结果。然而，逆图变算子的零模性质并不理想——尤其是它不与 b_0^- 对易。因此，我们使用截断到 NS-NS 和 RR 扇区 $\left(\mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-3/2,-3/2}\right)$ 的、对应弦场 $\widetilde{\Psi} \in \widetilde{\mathcal{H}}_c$ 的顶点算子来构造圆盘单点函数，并定义

$$\left\{\widetilde{A}^c\right\}_D = \Omega_0^{(0,1,1,0)}\left(\widetilde{A}^c\right) \equiv -\frac{1}{2\pi i} N_{0,1,1,0}\left\langle \widehat{\mathcal{G}}\widetilde{A}^c \left| c_0^- e^{-\Lambda(L_0+\bar{L}_0)} \right| B\right\rangle, \tag{158}$$

where $\Lambda \geq 0$ is an arbitrary parameter, that already appeared in (81) in the context of bosonic string amplitudes, $|B\rangle$ is the boundary state describing the D-brane, $N_{0,1,1,0}$ is a constant that has been given in (84), and

其中 $\Lambda \geq 0$ 是一个任意参数，它已经出现在玻色弦振幅背景下的 (81) 式中，$|B\rangle$ 是描述 D 膜的边界态，$N_{0,1,1,0}$ 是 (84) 式中已给出的常数，且

$$\widehat{\mathcal{G}} \equiv \begin{cases} 1 \text{ on } \mathcal{H}_{-1,-1}, \\ \frac{1}{2}\left(\mathcal{X}_0 + \overline{\mathcal{X}}_0\right) \text{ on } \mathcal{H}_{-3/2,-3/2}. \end{cases} \tag{159}$$

Note that the boundary state is a linear combination of states with picture numbers $(q, -2-q)$ for all integer and half-integer q , since for the disk one-point function of a closed string vertex operator, the picture number conservation only requires the sum of the left and right-handed picture numbers to be 2. Since the disk one-point functions of closed string states are proportional to the normalization constant K appearing in (12), the state $|B\rangle$ is proportional to K . As shown in (255), this in turn is proportional to the tension of the D-brane. The parameter Λ appearing in (158) reflects the freedom of scaling the local coordinate at the closed string puncture. Its value depends on the strategy used to produce string vertices that generate a cover of the moduli space.

请注意，边界态是所有整数和半整数 q 对应鬼数 $(q, -2-q)$ 的态的线性组合，因为对于闭弦顶点算子的圆盘单点函数，鬼数守恒仅要求左手性和右手性鬼数之和为 2。由于闭弦态的圆盘单点函数正比于 (12) 式中出现的归一化常数 K ，态 $|B\rangle$ 正比于 K 。如 (255) 式所示，这又正比于 D 膜的张力。(158) 式中出现的参数 Λ 反映了缩放闭弦穿刺点局部坐标的自由度。它的值取决于构造覆盖模空间的弦顶点所采用的方案。

For clarity, we note that for bosonic open-closed theory, the disk one-point function, going into the action, is

为清晰起见，我们说明: 对于玻色开-闭弦理论，进入作用量的圆盘单点函数为

$$\{A^c\}_D = \Omega_0^{(0,1,1,0)}(A^c) \equiv -\frac{1}{2\pi i} N_{0,1,1,0} \left\langle A^c \left| c_0^- e^{-\Lambda(L_0+\bar{L}_0)} \right| B \right\rangle. \tag{160}$$

We can also define multilinear products $[\cdots]_c$ and $[\cdots]_o$ via the relations [21]:

我们还可以通过关系定义多重线性积 $[\cdots]_c$ 和 $[\cdots]_o$ [21]:

$$\left\langle A_0^c \left| c_0^- \right| \left[A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\right]_c \right\rangle = \left\{A_0^c A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\right\}, \ \forall \left|A_0^c\right\rangle \in \mathcal{H}_c,$$

$$\left\langle A_0^o \left| \left[A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\right]_o \right| \right\rangle = \left\{A_1^c \cdots A_N^c; A_0^o A_1^o \cdots A_M^o\right\}, \ \forall \left|A_0^o\right\rangle \in \mathcal{H}_o,$$

$$\left\langle \widetilde{A}^c \left| c_0^- \right| []_c \right\rangle = \left\{\widetilde{A}^c\right\}_D, \tag{161}$$

where for convenience, we have chosen A_i^o ’ s to be even and A_i^o ’ s to be odd. Note that $[\cdots]_c$ is a closed string vertex operator, while $[\cdots]_o$ is an open string vertex operator. Following our earlier discussion, we can conclude that in $[\cdots]_c$, [is Grassmann odd, while in $[\cdots]_o$,[is Grassmann even. In $\{\cdots\}$, we do not have separate labels for closed and open strings, and { is always Grassmann even. The other useful information is that if $A_1^c, \cdots, A_N^c$ are Grassmann even and $A_1^o, \cdots, A_M^o$ are Grassmann odd, then $\left[A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\right]_c$ is Grassmann odd, and $\left[A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\right]_o$ is Grassmann even.

为方便起见，我们已设定 A_i^o 为格拉斯曼偶，A_i^o 为格拉斯曼奇。注意 $[\cdots]_c$ 是闭弦顶点算子，而 $[\cdots]_o$ 是开弦顶点算子。根据我们之前的讨论，可以得出结论: 在 $[\cdots]_c$ 中，[是格拉斯曼奇，而在 $[\cdots]_o$ 中，[是格拉斯曼偶。在 $\{\cdots\}$ 中，我们没有给闭弦和开弦分别标注，且 {始终为格拉斯曼偶。另一项有用的结论是: 若 $A_1^c, \cdots, A_N^c$ 为格拉斯曼偶，$A_1^o, \cdots, A_M^o$ 为格拉斯曼奇，则 $\left[A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\right]_c$ 为格拉斯曼奇，$\left[A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\right]_o$ 为格拉斯曼偶。

For Grassmann even $\widetilde{A}^c \in \widetilde{\mathcal{H}}_c, A_i^c \in \mathcal{H}_c$ and Grassmann odd $A_i^o \in \mathcal{H}_o$, the main identities take the form

对于格拉斯曼偶的 $\widetilde{A}^c \in \widetilde{\mathcal{H}}_c, A_i^c \in \mathcal{H}_c$ 和格拉斯曼奇的 $A_i^o \in \mathcal{H}_o$ ，主要恒等式形式为

$$\left\{\left(Q\widetilde{A}^c\right)\right\}_D = 0, \tag{162}$$

and

且

$$\begin{aligned}
&\sum_{i=1}^{N}\left\{A_1^c \cdots A_{i-1}^c\left(QA_i^c\right)A_{i+1}^c \cdots A_N^c; A_1^o \cdots A_M^o\right\} \\
&+\sum_{j=1}^{M}\left\{A_1^c \cdots A_N^c; A_1^o \cdots A_{j-1}^o\left(QA_j^o\right)A_{j+1}^o \cdots A_M^o\right\}(-1)^{j-1} \\
&= -\frac{1}{2}\sum_{k=0}^{N}\sum_{\substack{\ell=0 \\ \{j_1,\cdots,i_k\}\subset\{1,\cdots,N\}}}^{M}\left(\left\{A_{i_1}^c \cdots A_{i_k}^c \mathcal{B}^c; A_{j_1}^o \cdots A_{j_\ell}^o\right\}\right. \\
&\qquad \left.+\left\{A_{i_1}^c \cdots A_{i_k}^c; \mathcal{B}^o A_{j_1}^o \cdots A_{j_\ell}^o\right\}\right) \\
&\qquad -\left\{\left[A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\right]_c\right\}_D \\
&\qquad -\frac{1}{2}\left\{A_1^c \cdots A_N^c \varphi_s \varphi_r; A_1^o \cdots, A_M^o\right\}\langle\varphi_s^c, \mathcal{G}\varphi_r^c\rangle \\
&\qquad -\frac{1}{2}(-1)^{\widehat{\varphi}_s}\left\{A_1^c \cdots A_N^c; \widehat{\varphi}_s\widehat{\varphi}_r A_1^o \cdots, A_M^o\right\}\langle\widehat{\varphi}_s^c, \mathcal{G}\widehat{\varphi}_r^c\rangle .
\end{aligned} \tag{163}$$

In here, $\mathcal{B}^c$ and $\mathcal{B}^o$ are open and closed string states, respectively, given by

此处，$\mathcal{B}^c$ 和 $\mathcal{B}^o$ 分别为开弦态和闭弦态，由下式给出

$$\begin{aligned}
&\mathcal{B}^c \equiv \mathcal{G}\left[A_{\bar{i}_1}^c \cdots A_{\bar{i}_{N-k}}^c; A_{\bar{j}_1}^o \cdots A_{\bar{j}_{M-\ell}}^o\right]_c,\ \mathcal{B}^o \equiv \mathcal{G}\left[A_{\bar{i}_1}^c \cdots A_{\bar{i}_{N-k}}^c; A_{\bar{j}_1}^o \cdots A_{\bar{j}_{M-\ell}}^o\right]_o, \\
&\{i_1, \cdots, i_k\} \cup \left\{\bar{i}_1, \cdots, \bar{i}_{N-k}\right\} = \{1, \cdots, N\}, \{j_1, \cdots, j_\ell\} \cup \left\{\bar{j}_1, \cdots, \bar{j}_{M-\ell}\right\} \\
&= \{1, \cdots, M\},
\end{aligned}$$

(164)

and $\{\widehat{\varphi}_s\}$ and $\{\widehat{\varphi}^c_s\}$ each represent complete basis of open string states satisfying condition similar to (111), and $(-1)^{\widehat{\varphi}_s}$ represents Grassmann parity of the vertex operator $\widehat{\varphi}_s$. For other choices of Grassmann parities of A^c_i or A^c_j , we first multiply the odd A^c_i ' s and even A^o_i ' s by Grassmann odd elements so that (163) holds and then move these Grassmann odd parameters to the extreme left on both sides of the equation using the rules described earlier.

且 $\{\widehat{\varphi}_s\}$ 与 $\{\widehat{\varphi}^c_s\}$ 各自代表满足与 (111) 类似条件的开弦态的完全基，$(-1)^{\widehat{\varphi}_s}$ 代表顶点算符 $\widehat{\varphi}_s$ 的格拉斯曼奇偶性。对于 A^c_i 或 A^c_j 格拉斯曼奇偶性的其他选择，我们先将奇数性的 A^c_i 和偶数性的 A^o_i 乘上格拉斯曼奇元素，使 (163) 式成立，再利用前文所述规则将这些格拉斯曼奇参数移到方程两侧的最左端。

We are now in a position to write down the BV master action of open-closed string field theory action. It is given by

我们现在可以写出开-闭弦场论的 BV 主作用量了，它由下式给出

$$S = -\frac{1}{2}\left\langle\widetilde{\Psi}_c, Q\mathcal{G}\widetilde{\Psi}_c\right\rangle + \left\langle\widetilde{\Psi}_c, Q\Psi_c\right\rangle - \frac{1}{2}\left\langle\widetilde{\Psi}_o, Q\mathcal{G}\widetilde{\Psi}_o\right\rangle + \left\langle\widetilde{\Psi}_o, Q\Psi_o\right\rangle$$

$$+\left\{\widetilde{\Psi}_c\right\}_D + \sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{N!M!}\left\{(\Psi_c)^N;(\Psi_o)^M\right\}. \tag{165}$$

The action (165) satisfies the BV master equation with the following definition of the antibracket. If for an arbitrary function $F\left(\Psi_c, \widetilde{\Psi}_c, \Psi^o, \widetilde{\Psi}^o\right)$, one expresses the first-order variation as

作用量 (165) 在反括号的下述定义下满足 BV 主方程。若对任意函数 $F\left(\Psi_c, \widetilde{\Psi}_c, \Psi^o, \widetilde{\Psi}^o\right)$，我们将一阶变分写为

$$\delta F = \left\langle F^c_R, \delta\widetilde{\Psi}_c\right\rangle + \left\langle \widetilde{F}^c_R, \delta\Psi_c\right\rangle + \left\langle F^o_R, \delta\widetilde{\Psi}_o\right\rangle + \left\langle \widetilde{F}^o_R, \delta\Psi_o\right\rangle$$

$$= \left\langle \delta\widetilde{\Psi}_c, F^c_L\right\rangle + \left\langle \delta\Psi_c, \widetilde{F}^c_L\right\rangle + \left\langle \delta\widetilde{\Psi}_o, F^o_L\right\rangle + \left\langle \delta\Psi_o, \widetilde{F}^o_L\right\rangle, \tag{166}$$

then the antibracket of F and G is defined as

则 F 和 G 的反括号定义为

$$\{F, G\} = -\left(\left\langle F^c_R, \widetilde{G}^c_L\right\rangle + \left\langle \widetilde{F}^c_R, G^c_L\right\rangle + \left\langle \widetilde{F}^c_R, \mathcal{G}\widetilde{G}^c_L\right\rangle\right)$$

$$-\left(\left\langle F^o_R, \widetilde{G}^o_L\right\rangle + \left\langle \widetilde{F}^o_R, G^o_L\right\rangle + \left\langle \widetilde{F}^o_R, \mathcal{G}\widetilde{G}^o_L\right\rangle\right). \tag{167}$$

For the record, we shall also write down the open-closed bosonic string field theory action, obtained by setting $\widetilde{\Psi}$ to Ψ and $\mathcal{G}$ to 1 in (165):

作为记录，我们也写出开-闭玻色弦场论的作用量，它由在 (165) 中将 $\widetilde{\Psi}$ 设为 Ψ 、$\mathcal{G}$ 设为 1 得到:

$$S = \frac{1}{2}\langle\Psi_c, Q\Psi_c\rangle + \frac{1}{2}\langle\Psi_o, Q\Psi_o\rangle + \{\Psi_c\}_D + \sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{N!M!}\left\{(\Psi_c)^N; (\Psi_o)^M\right\}.$$

(168)

In the open-closed string field theory action, one can ask which terms belong to the classical theory and which terms are to be considered quantum. This is obtained by taking the $g_s \to 0$ limit with some combination of the fields and g_s fixed and then identifying the leading order terms in the action as the classical action. The limit clearly depends on which combination of fields we keep fixed as we take the $g_s \to 0$ limit. It is reasonable to take the limit in a way so as to include the kinetic term of open strings and the open string interactions on disks in the classical action; this is after all, the classical open string field theory. We also desire that the closed string kinetic term should be part of the classical action since it is the sum of these two BRST operators, open and closed, that acts as a boundary operator in moduli space and the two of them should appear at the same order in a g_s expansion. To this end, let us define new string fields $\chi_c = g_s^{1/2}\Psi_c, \chi_o = g_s^{1/2}\Psi_o$ and take the small g_s limit keeping χ_c and χ_o fixed. In this case, the kinetic terms of both the open and the closed string will acquire factors of g_s^{-1} , and an interaction term in the action proportional to $\chi_c^{n_c}\chi_o^{n_o}$ will get an additional factor of $g_s^{-(n_c+n_o)/2}$. Using (157), we see that the net power of g_s in an interaction term is given by g_s^{p-1} with $p = 2g + b + \frac{n_c}{2} - 1$. The leading terms have $p = 0$, and these terms appear in the action with an overall factor of $1/g_s$. Since this is the same order as the kinetic terms, we can interpret them as part of the classical action, with the overall $1/g_s$ factor playing the role of $1/\hbar$ factor that usually accompanies the classical action. It is clear that $p > 0$ unless $g = 0, b = 1$, and $n_c = 0$. So none of the interactions, except for open strings on a disk, are part of the classical open-closed string field theory. The classical open string field theory and free closed strings form the classical open-closed string field theory [7].

在开-闭弦场论作用量中，我们可以区分哪些项属于经典理论，哪些项需要被视为量子项。该区分通过如下方式得到: 取 $g_s \to 0$ 极限，固定场与 g_s 的某种组合，随后将作用量中的领头阶项识别为经典作用量。该极限显然依赖于我们取 $g_s \to 0$ 极限时固定哪一种场组合。合理的取极限方式需要将开弦动能项与圆盘上的开弦相互作用纳入经典作用量中；这毕竟就是经典开弦场论。我们也要求闭弦动能项属于经典作用量，因为作为模空间中的边界算子的是开、闭 BRST 算符之和，二者在 g_s 展开中应当出现在同一阶。为此，我们定义新弦场 $\chi_c = g_s^{1/2}\Psi_c, \chi_o = g_s^{1/2}\Psi_o$ ，固定 χ_c 和 χ_o 取小 g_s 极限。在此情况下，开弦与闭弦的动能项都会获得因子 g_s^{-1} ，作用量中正比于 $\chi_c^{n_c}\chi_o^{n_o}$ 的相互作用项会得到额外因子 $g_s^{-(n_c+n_o)/2}$ 。利用 (157) 我们可以看到，相互作用项中 g_s 的总幂次由 g_s^{p-1} 给出，其中 $p = 2g + b + \frac{n_c}{2} - 1$ 。领头项满足 $p = 0$ ，这些项在作用量中整体带有因子 g_s 。由于这和动能项的阶数相同，我们可以将它们解释为经典作用量的一部分，整体因子 $1/g_s$ 扮演着通常伴随经典作用量的 $1/\hbar$ 因子的角色。显然，除非满足 $p > 0$ 除非 $g = 0, b = 1$ 和 $n_c = 0$ ，否则不成立。因此除了圆盘上的开弦相互作用外，没有其他相互作用属于经典开-闭弦场论。经典开弦场论加上自由闭弦就构成了经典开-闭弦场论 [7]。

Before ending this section, we shall discuss the relation between the open-closed string field theory discussed here and the tree-level open string field theory action (139). From the bosonic version of the open-closed SFT action (168), restricted to tree-level open string theory, we have

在结束本节之前，我们将讨论本文介绍的开-闭弦场论与树级开弦场论作用量 (139) 之间的关系。对玻色型开-闭弦场论作用量 (168) 做树级开弦理论限制，我们得到

$$
\begin{aligned}
S_{oc} &= \frac{1}{2}\langle\Psi_o, Q\Psi_o\rangle + \sum_{n=3}^{\infty}\frac{1}{n!}g_s^{(n-2)/2}\int \Omega_{n-3}^{(0,1,0,n)}(\Psi_o,\cdots,\Psi_o) \\
&= \frac{1}{2}K\langle\Psi_o, Q\Psi_o\rangle' + \sum_{n=3}^{\infty}\frac{1}{n!}g_s^{(n-2)/2}N_{0,1,0,n}K\int \Omega_{n-3}^{o(0,n)}(\Psi_o,\cdots,\Psi_o),
\end{aligned}
$$

(169)

where in the last step we used (69) and (136). On the other hand, the classical open string field theory action given in (138), (139) gives

其中最后一步我们用到了 (69) 和 (136)。另一方面，(138)、(139) 中给出的经典开弦场论作用量给出

$$
S_o = \frac{1}{2}\langle\psi_o, Q\psi_o\rangle' + \sum_{n=3}^{\infty}\frac{1}{n!}g_o^{n-2}\int \Omega_{n-3}^{o(0,n)}(\psi_o,\cdots,\psi_o). \tag{170}
$$

To compare the actions, we let

为了比较两个作用量，我们令

$$
\Psi_o = K^{-1/2}\psi_o \tag{171}
$$

in (169) and use (84) to get

代入 (169) 并利用 (84) 得到

$$
S_{oc} = \frac{1}{2}\langle\psi_o, Q\psi_o\rangle' + \sum_{n=3}^{\infty}\frac{1}{n!}g_s^{(n-2)/2}K^{(2-n)/2}\eta_c^{3(n-2)/4}\int \Omega_{n-3}^{o(0,n)}(\psi_o,\cdots,\psi_o).
$$

(172)

With the kinetic terms of the open SFT and open-closed SFT now matching, we compare the interaction terms in (170) and (172) and see that they agree if we take

开弦场论与开-闭弦场论的动能项匹配之后，我们比较 (170) 和 (172) 中的相互作用项，可以看到当取如下关系时二者一致

$$
g_o = g_s^{1/2}K^{-1/2}\eta_c^{3/4}. \tag{173}
$$

Squaring, we get

平方后我们得到

$$
g_o^2 = (g_s/K)\,\eta_c^{3/2}. \tag{174}
$$

This is the relation between the open and closed string field theory couplings. In section "Results for Special Amplitudes", we shall relate K to the D-brane tension.

这就是开弦场论耦合与闭弦场论耦合之间的关系。在“特殊振幅的结果”一节中，我们会将 K 与 D 膜张力联系起来。

Kinetic Term for Massless Open String Fields

无质量开弦场的动能项

For the conventional classical action, we restrict the sum over states in the expansion of $|\psi_o\rangle$ to those of ghost number one. That action inherits the gauge invariance of the master action, when we consider a ghost number zero string field gauge parameter $|\varepsilon\rangle$. We now illustrate the basics of the bosonic open string field theory by evaluating the kinetic term for the massless sector of the open string field theory formulated on the BCFT that represents the (unstable) space-filling D-brane.

对于传统经典作用量，我们将 $|\psi_o\rangle$ 展开中的态求和限制为鬼数为 1 的态。当我们取鬼数为 0 的弦场规范参数 $|\varepsilon\rangle$ 时，该作用量继承了主作用量的规范不变性。我们接下来通过计算表述在代表 (不稳定) 空间填充 D 膜的 BCFT 上的开弦场论无质量 sector 的动能项，展示玻色开弦场论的基本内容。

For the purpose of illustration, we shall keep the factors of α' in this and the next subsection, setting $\alpha' = 1$ elsewhere (unless explicitly noted). For this,

为了便于说明，我们将在本小节和下一小节保留 α' 因子，在其他地方令 $\alpha' = 1$ (除非另有明确说明)。据此，

The coupling constants g_o and g_s are taken to be unit - free pure numbers.

耦合常数 g_o 和 g_s 被取为无单位的纯数。

(175)

Moreover, for simplicity, we declare the string fields $\left|\Psi\rangle = \sum_i \right| \varphi_i\rangle\psi^i$, to also be unit-free, with both the component target space fields ψ^i and the basis states $|\varphi_i\rangle$ unit-free:

此外，为简化起见，我们约定弦场 $\left|\Psi\rangle = \sum_i \right| \varphi_i\rangle\psi^i$ 也无单位，分量目标空间场 ψ^i 和基态 $|\varphi_i\rangle$ 均无单位:

The string field $|\Psi\rangle$, basis states $|\varphi_i\rangle$, and target space fields ψ^i are all unit free.

弦场 $|\Psi\rangle$ 、基态 $|\varphi_i\rangle$ 和目标空间场 ψ^i 全都无单位。

(176)

Note that the BRST operator Q is also unit-free, and with these conventions, the string field theory actions are unit-free. In order to incorporate α' factors, we need to accompany every power of momentum by $(\alpha')^{1/2}$ and every power of coordinate by $(\alpha')^{-1/2}$.

注意 BRST 算符 Q 也是无单位的，在这些约定下，弦场论作用量都是无单位的。为了引入 α' 因子，我们需要对每一阶动量配 $(\alpha')^{1/2}$ ，对每一阶坐标配 $(\alpha')^{-1/2}$ 。

For the massless sector, the string field is constructed with the following requirements. We define the number operator $\widehat{N}$ to be the one that counts the L_0 eigenvalues of matter and ghost oscillators and take a string field with zero $\widehat{N}$ eigenvalue. Moreover, the state must have ghost number one. Recalling that $c_n\,|0\rangle = 0$ for $n \geq 2$ and $b_n|\,0\rangle = 0$ for $n \geq -1$, we have, on a D- $(d-1)$ brane,

对于无质量 sector，弦场按以下要求构造。我们定义数算符 $\widehat{N}$ 为计数物质和鬼振子 L_0 本征值的算符，取弦场的 $\widehat{N}$ 本征值为零。此外，态的鬼数必须为 1。回忆 $c_n\,|0\rangle = 0$ for $n \geq 2$ and $b_n|\,0\rangle = 0$ 对应 $n \geq -1$ ，我们可知在 D- $(d-1)$ 膜上，

$$|\psi_o\rangle = (\alpha')^{(p+1)/2} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \left(A_\mu(k)\, c_1 \alpha_{-1}^{\mu} - i\sqrt{\frac{1}{2}} B(k)\, c_0 \right) |k\rangle, \tag{177}$$

with $|k\rangle = e^{ik\cdot X}|\,0\rangle$ and α_{-n} being the oscillators of $i\sqrt{\frac{2}{\alpha'}}\partial X$. We have two fields here, a gauge field A_μ and an auxiliary scalar field B (both unit-free). No antighost oscillator can appear here, because it would have to be a b_{-2} carrying number two or an oscillator with an even larger number, and if present, we cannot achieve zero number, since only c_1 reduces the number by one, but it cannot appear twice. Linearized gauge transformations take the form $\delta\,|\psi_o\rangle = Q|\varepsilon\rangle$, with $|\varepsilon\rangle$ a ghost number zero field, also with total number $\widehat{N} = 0$. There is just one such state

其中 $|k\rangle = e^{ik\cdot X}|\,0\rangle$ 和 α_{-n} 是 $i\sqrt{\frac{2}{\alpha'}}\partial X$ 的振子。我们这里得到两个场: 规范场 A_μ 和辅助标量场 B (二者均无单位)。这里不能出现反鬼振子，因为反鬼振子要么是携带数为 2 的 b_{-2} ，要么是数更大的振子；如果反鬼振子存在，我们无法得到零总个数——只有 c_1 能让总个数减一，但它不能出现两次。线性化规范变换形如 $\delta\,|\psi_o\rangle = Q|\varepsilon\rangle$ ，其中 $|\varepsilon\rangle$ 是鬼数为零的场，总个数也为 $\widehat{N} = 0$ 。这样的态仅有一个

$$\Big|\varepsilon\rangle = (\alpha')^{(p+1)/2} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \frac{i}{\sqrt{2}} \varepsilon(k) \Big| k\rangle. \tag{178}$$

The signs have been included in the string field and gauge parameter for convenience. When passing from momentum to coordinate space, our conventions for Fourier transformation are

我们为了方便已经将符号纳入弦场和规范参数中。从动量空间转换到坐标空间时，我们对傅里叶变换的约定为

$$\phi(x) = (\alpha')^{(p+1)/2} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \widetilde{\phi}(k)\, e^{ikx}, \quad \text{so that } ik_\mu \leftrightarrow \partial_\mu, \tag{179}$$

allowing for both ϕ and $\widetilde{\phi}$ to be unit-free. To calculate the action, we need the BPZ dual of the string field. We first note that the BPZ dual of $|k\rangle$ is $\langle k|$. Moreover, for any oscillator ϕ_n arising from a dimension d conformal field $\phi = \sum_n \frac{\phi_n}{z^{n+d}}$, its BPZ dual is

使得 ϕ 和 $\widetilde{\phi}$ 都为无量纲。为计算作用量，我们需要弦场的 BPZ 对偶。首先注意 $|k\rangle$ is $\langle k|$ 的 BPZ 对偶。此外，对任意来自维度 d 共形场 $\phi=\sum_{n}\frac{\phi_n}{z^{n+d}}$ 的振荡器 ϕ_n，其 BPZ 对偶为

$$\operatorname{bpz}(\phi_n)=(-1)^{n+d}\phi_{-n}. \tag{180}$$

This result follows from the conformal transformation implementing the BPZ map: $z\to-1/z$ which maps the infinite past $z=0$ to the infinite future $z\to\infty$ in the UHP. All in all, the BPZ dual string field is found to be

该结果可由实现 BPZ 映射的共形变换得到: $z\to-1/z$ 将上半平面的无穷远过去 $z=0$ 映射到无穷远未来 $z\to\infty$。综上，最终得到 BPZ 对偶弦场为

$$\langle\psi_o|=(\alpha')^{(p+1)/2}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}\left\langle k\right|\left(A_\mu(k)c_{-1}\alpha_1^\mu+i\sqrt{\frac{1}{2}}B(k)c_0\right), \tag{181}$$

The computation below requires the BRST operator (1), which in oscillator form is given by:

下述计算需要用到 BRST 算符 (1)，其振荡器形式由下式给出:

$$Q=\sum_n c_n\left(L_{-n}^m-\delta_{n,0}\right)+\frac{1}{2}\sum_{m,n}(m-n):c_mc_nb_{-n-m}:, \tag{182}$$

where :: denotes normal ordering in which all the oscillators carrying positive mode number (negative eigenvalue of the number operator) are placed to the right of all the oscillators carrying negative mode number (positive eigenvalue of the number operator). Here the matter Virasoro operators are

其中:: 代表正规序，所有正模编号的振荡器 (对应数算符的负本征值) 都放在所有负模编号的振荡器 (对应数算符的正本征值) 的右侧。此处物质 Virasoro 算符为

$$L_n^m=\frac{1}{2}\sum_m\alpha_m\cdot\alpha_{n-m},(n\neq0),\ L_0^m=\frac{1}{2}\alpha_0^2+\sum_{n\geq1}\alpha_{-n}\cdot\alpha_n,\ \alpha_0^\mu=\sqrt{2\alpha'}k^\mu.$$

(183)

We have a ”level expansion” of the BRST operator, starting with level zero and going up. In that expansion, a term of level n is one where the labels of the positively moded oscillators add up to n (while the negatively moded labels add up to $-n$, of course). We have

我们对 BRST 算符做“能级展开”，从零能级开始逐步向上。在该展开中，n 能级的项满足所有正模振荡器的编号之和为 n (当然，负模振荡器的编号之和为 $-n$)。我们得到

$$Q=c_0\left(\alpha'k^2-1\right)$$

$$+c_0\left(\alpha_{-1}\cdot\alpha_1+b_{-1}c_1+c_{-1}b_1\right)$$

$$+\sqrt{2\alpha'}k_\mu\left(\alpha_{-1}^\mu c_1 + c_{-1}\alpha_1^\mu\right) - 2b_0c_{-1}c_1 \tag{184}$$

$$+...$$

The first line above is the level zero contribution to Q , and the next two lines contain level one contributions. Higher-level terms in Q , represented by the dots, will kill the string field we are using. We have the following action of the BRST operator on states:

上面第一行是对 Q 的零能级贡献，后两行包含一级贡献。省略号代表 Q 中更高能级的项，这些项会消除我们所用的弦场。BRST 算符对态的作用如下:

$$Qc_1\alpha_{-1}^\mu\left|k\right\rangle = \left(\alpha' k^2 c_0c_1\alpha_{-1}^\mu + \sqrt{2\alpha'}k^\mu c_{-1}c_1\right)\left|k\right\rangle$$

$$Qc_0\left|k\right\rangle = \left(\sqrt{2\alpha'}k_\mu\alpha_{-1}^\mu c_1c_0 - 2c_{-1}c_1\right)\left|k\right\rangle, \tag{185}$$

$$Q\left|k\right\rangle = \left(\alpha' k^2 c_0 + \sqrt{2\alpha'}k_\mu\alpha_{-1}^\mu c_1\right)\left|k\right\rangle.$$

The basic correlator (11) with the proper factor of α' inserted reads:

插入正确因子 α' 后的基本关联式 (11) 为:

$$\langle k\left|c_{-1}c_0c_1\right|k'\rangle' = -(2\pi)^{p+1}(\alpha')^{-(p+1)/2}\delta^{(p+1)}(k+k'). \tag{186}$$

This correlator enables us to compute the free action, the first term in (139). Using the above action of Q , a bit of calculation gives

该关联式可用于计算自由作用量，即 (139) 中的第一项。利用上述 Q 的作用，经过少量计算可得

$$S_2 = \frac{1}{2}\langle\psi_o\left|Q\right|\psi_o\rangle'$$

$$= (\alpha')^{(p+1)/2}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}\left(-\frac{1}{2}A^\mu(-k)\,\alpha' p^2A_\mu(k) - \sqrt{\alpha'}A^\mu(-k)\,ik_\mu B(k)\right. \tag{B k}$$

$$\left.-\frac{1}{2}B(-k)\,B(k)\right).$$

(187)

In coordinate space, this gives

在坐标空间中，结果为

$$S_2 = (\alpha')^{-(p+1)/2}\int d^{p+1}x\left(\frac{\alpha'}{2}A^\mu\Box A_\mu - \sqrt{\alpha'}A^\mu\partial_\mu B - \frac{1}{2}B^2\right). \tag{188}$$

Linearized gauge transformations take the form $\delta\left|\psi_o\right\rangle = Q|\varepsilon\rangle$ and give

线性化规范变换形如 $\delta|\psi_o\rangle = Q|\varepsilon\rangle$ ，给出

$$\delta A_\mu(k) = i\sqrt{\alpha'}k_\mu\varepsilon(k),\ \delta B(k) = -\alpha' k^2\varepsilon(k). \tag{189}$$

In coordinate space, we have

在坐标空间中，我们有

$$\delta A_\mu = \sqrt{\alpha'}\partial_\mu\varepsilon,\ \delta B = \alpha'\Box\varepsilon. \tag{190}$$

It is quickly verified that gauge transformations leave the action invariant (up to total derivatives). Elimination of the auxiliary field B via its equation of motion gives $B = \sqrt{\alpha'}\partial\cdot A$, and we then get

可以很快验证规范变换使作用量保持不变 (差全导数项)。通过运动方程消去辅助场 B 得到 $B = \sqrt{\alpha'}\partial\cdot A$ ，随后我们得到

$$\begin{aligned} S &= (\alpha')^{-(p-1)/2}\int d^{p+1}x\left(\frac{1}{2}A_\mu\Box A^\mu + \frac{1}{2}(\partial\cdot A)^2\right) \\ &= (\alpha')^{-(p-1)/2}\int d^{p+1}x\left(-\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu\right) \\ &= (\alpha')^{-(p-1)/2}\int d^{p+1}x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right), \end{aligned} \tag{191}$$

which, with $F_{\mu\nu}\equiv\partial_\mu A_\nu - \partial_\nu A_\mu$, is the familiar free gauge-invariant action for a Maxwell field. The gauge transformation is $\delta A_\mu = \sqrt{\alpha'}\partial_\mu\varepsilon$.

代入 $F_{\mu\nu}\equiv\partial_\mu A_\nu - \partial_\nu A_\mu$ 后，这就是麦克斯韦场我们熟悉的自由规范不变作用量。规范变换为 $\delta A_\mu = \sqrt{\alpha'}\partial_\mu\varepsilon$ 。

Kinetic Term for Massless Closed String Fields

无质量闭弦场的动力学项

We shall begin with relating the coefficients of expansion of the string field with the degrees of freedom of the metric. For this, we observe that in the presence of a background target space string metric $g_{\mu\nu}$, the world-sheet action S_{WS} associated with the non-compact coordinates takes the form

我们首先将弦场的展开系数与度规的自由度联系起来。为此，我们注意到，当存在背景靶空间弦度规 $g_{\mu\nu}$ 时，与非紧致坐标相关的世界面作用量 S_{WS} 形式为

$$S_{\text{ws}} = -\frac{1}{4\pi\alpha'}\int dxdy g_{\mu\nu}\left(\partial_x X^\mu\partial_x X^\nu + \partial_y X^\mu\partial_y X^\nu\right)$$

$$= -\frac{1}{\pi\alpha'}\int dxdy g_{\mu\nu}\partial X^\mu \overline{\partial} X^\nu,\ z \equiv x + iy. \tag{192}$$

with x, y real coordinates on the world-sheet and ∂ and $\overline{\partial}$ derivatives with respect to z and $\bar{z}$, respectively. Writing

其中 x, y 是世界面上的实坐标，∂ 和 $\overline{\partial}$ 分别是对 z 和 $\bar{z}$ 的导数。记

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{193}$$

we see that to first order in $h_{\mu\nu}$, the deformation by $h_{\mu\nu}$ corresponds to the insertion of the operator

我们可以看到，在 $h_{\mu\nu}$ 的一阶近似下，由 $h_{\mu\nu}$ 给出的形变对应插入算符

$$S_{\text{ws}}|_h = -\frac{1}{\pi\alpha'}\int dxdy h_{\mu\nu}\partial X^\mu \overline{\partial} X^\nu, \tag{194}$$

in the world-sheet correlator. On the other hand, it follows from (51) and (107) that if we deform the background string field by $\mathcal{O} = c\bar{c}V$ for some dimension (1,1) matter primary V , it corresponds to inserting into the world-sheet correlator a term

插入到世界面关联函数中。另一方面，由式 (51) 和 (107) 可知，若我们针对某个维度为 (1,1) 的物质主量场 V ，将背景弦场形变 $\mathcal{O} = c\bar{c}V$ ，这对应向世界面关联函数中插入一项

$$-g_s\frac{1}{\pi}\int dxdyV \tag{195}$$

From the last two equations, we can read V and see that in order to turn on a metric deformation $h_{\mu\nu}$, we need to turn on a string field background $\mathcal{O}_h = ccV$ given by

从上述两个方程中，我们可以读出 V ，可知要开启度规形变 $h_{\mu\nu}$ ，我们需要开启弦场背景 $\mathcal{O}_h = ccV$ ，其表达式为

$$\mathcal{O}_h = \frac{1}{g_s\alpha'}h_{\mu\nu}c\bar{c}\partial X^\mu \overline{\partial} X^\nu = \frac{1}{g_s}\left(-\frac{1}{2}h_{\mu\nu}\right)c\bar{c}i\sqrt{\frac{2}{\alpha'}}\partial X^\mu i\sqrt{\frac{2}{\alpha'}}\overline{\partial} X^\nu. \tag{196}$$

Recalling that α_{-n} are the oscillators of $i\sqrt{\frac{2}{\alpha'}}\partial X$, and similarly for the barred oscillators, this suggests that we take the closed string field for the massless bosonic fields is given as

回顾 α_{-n} 是 $i\sqrt{\frac{2}{\alpha'}}\partial X$ 的振荡器，带横杠的振荡器也类似，这说明无质量玻色场的闭弦场形式为

$$|\Psi\rangle = (\alpha')^{D/2}\frac{1}{g_s}\int\frac{d^Dk}{(2\pi)^D}\Big(-\frac{1}{2}e_{\mu\nu}(k)\,\alpha_{-1}^\mu\overline{\alpha}_{-1}^\nu c_1\bar{c}_1 + e(k)\,c_1c_{-1} + \bar{e}(k)\,\bar{c}_1\bar{c}_{-1}$$
$$+i\sqrt{\frac{1}{2}}\left(f_\mu(k)\,c_0^+c_1\alpha_{-1}^\mu + \bar{f}_\mu(k)\,c_0^+\bar{c}_1\overline{\alpha}_{-1}^\mu\right)\Big)|k\rangle.$$

(197)

Here we can identify $h_{\mu\nu}$ as the symmetric part of $e_{\mu\nu}$ and the antisymmetric tensor field $b_{\mu\nu}$ as the antisymmetric part of $e_{\mu\nu}$:

此处我们可以将 $h_{\mu\nu}$ 识别为 $e_{\mu\nu}$ 的对称部分，将反对称张量场 $b_{\mu\nu}$ 识别为 $e_{\mu\nu}$ 的反对称部分:

$$e_{\mu\upsilon} = h_{\mu\upsilon} + b_{\mu\upsilon}, \ \text{ with } \ h_{\mu\upsilon} = h_{\upsilon\mu}, \ b_{\mu\upsilon} = -b_{\upsilon\mu}. \tag{198}$$

The above string field has $\widehat{N} = \widehat{\bar{N}} = 0$, where both these number operators are defined so that $|p\rangle$ has number zero. It follows that the level-matching constraint $L_0 - \bar{L}_0 = 0$ is satisfied. Moreover, $c_0^{\pm} \equiv \frac{1}{2}(c_0 \pm \bar{c}_0)$, and we also define $b_0^{\pm} \equiv \ b_0 \pm \bar{b}_0$, so that the non-vanishing anticommutators are $\left\{c_0^{\pm}, b_0^{\pm}\right\} = 1$. As required, $b_0^{-}|\Psi\rangle = 0$ because $b_0^{-}|\,k\rangle = 0$ and the ghost oscillator c_0^{-} does not appear in $|\Psi\rangle$. This expansion of the string field features five momentum-space component fields: $e_{\mu\upsilon}, e, \bar{e}, f$, and $\bar{f}$. In terms of the $\pm$ ghost zero modes, the basic correlator (10) becomes:

上述弦场满足 $\widehat{N} = \widehat{\bar{N}} = 0$ ，其中这两个数算符的定义都使得 $|p\rangle$ 的数为零。由此能级匹配约束 $L_0 - \bar{L}_0 = 0$ 得到满足。此外，$c_0^{\pm} \equiv \frac{1}{2}(c_0 \pm \bar{c}_0)$ ，我们还定义 $b_0^{\pm} \equiv \ b_0 \pm \bar{b}_0$ ，因此非零反对易子为 $\left\{c_0^{\pm}, b_0^{\pm}\right\} = 1$ 。符合要求的是，$b_0^{-}|\Psi\rangle = 0$ because $b_0^{-}|\,k\rangle = 0$ ，鬼振荡器 c_0^{-} 不出现在 $|\Psi\rangle$ 中。该弦场展开包含五个动量空间分量场: $e_{\mu\upsilon}, e, \bar{e}, f$ 和 $\bar{f}$ 。用 $\pm$ 鬼零模表示，基本关联函数 (10) 变为:

$$\left\langle k\left|c_{-1}\bar{c}_{-1}c_0^{-}c_0^{+}c_1\bar{c}_1\right|k'\right\rangle = -\frac{1}{2}(\alpha')^{-D/2}(2\pi)^D\delta^{(D)}(k+k'). \tag{199}$$

For closed strings, BPZ conjugation can be implemented with the map $z \to 1/z$ (without the minus signs of open strings), and the result is that for the oscillators of a dimension d field, we have [6]

对于闭弦，BPZ 共轭可以通过映射 $z \to 1/z$ 实现 (没有开弦的负号)，对于维度为 d 的场的振荡器，结果为 [6]

$$\mathrm{bpz}(\phi_n) = (-1)^d\phi_{-n}, \tag{200}$$

all oscillators transforming with the same sign prefactor, independent of the mode number. It then follows that BPZ conjugate of the above string field, needed for the construction of the kinetic term in the action, is given by

所有振荡器变换时都带有相同的符号前置因子，与模序数无关。由此可得，构造作用量中动力学项所需的上述弦场的 BPZ 共轭为

$$\begin{aligned}\langle\Psi| = (\alpha')^{D/2}\frac{1}{g_s}\int\frac{d^Dk}{(2\pi)^D}\langle k|\Big(&-\frac{1}{2}e_{\mu\upsilon}(k)\,\alpha_1^{\mu}\bar{\alpha}_1^{\upsilon}c_{-1}\bar{c}_{-1} + e(k)\,c_{-1}c_1 \\ &+\bar{e}(k)\,\bar{c}_{-1}\bar{c}_1 - i\sqrt{\frac{1}{2}}\left(f_{\mu}(k)\,c_0^{+}c_{-1}\alpha_1^{\mu} + \bar{f}_{\mu}(k)\,c_0^{+}\bar{c}_{-1}\bar{\alpha}_1^{\mu}\right).\end{aligned} \tag{201}$$

We wish to construct the quadratic term of the bosonic string action (114), given by

我们希望构造玻色弦作用量 (114) 的二次项，其形式为

$$S^{(2)} = \frac{1}{2} \langle \Psi | c_0^- Q | \Psi \rangle . \tag{202}$$

Here Q is the (ghost-number one) BRST operator of the conformal field theory. The BRST operator (1) is the sum of a holomorphic part that we used for the open bosonic string in (182) and (183) and the analogous antiholomorphic part. This time, however, $\alpha_0 = \overline{\alpha}_0 = \sqrt{\frac{\alpha'}{2}} k$. The level expansion of the BRST operator now gives:

此处 Q 是共形场论的 (鬼数为 1 的)BRST 算符。BRST 算符 (1) 由两部分相加得到: 一部分是我们在式 (182) 和 (183) 中用于开玻色弦的全纯部分，另一部分是对应的反全纯部分。但这一次，$\alpha_0 = \overline{\alpha}_0 = \sqrt{\frac{\alpha'}{2}} k$ 。BRST 算符的能级展开如下:

$$Q = c_0^+ \left(\frac{\alpha'}{2} k^2 - 2 \right)$$

$$+ c_0^+ \left(\alpha_{-1} \cdot \alpha_1 + b_{-1} c_1 + c_{-1} b_1 + \overline{\alpha}_{-1} \cdot \overline{\alpha}_1 + \bar{b}_{-1} \bar{c}_1 + \bar{c}_{-1} \bar{b}_1 \right) \tag{203}$$

$$+ \sqrt{\frac{\alpha'}{2}} k \cdot \left(\alpha_{-1} c_1 + c_{-1} \alpha_1 \right) + \sqrt{\frac{\alpha'}{2}} k \cdot \left(\overline{\alpha}_{-1} \bar{c}_1 + \bar{c}_{-1} \overline{\alpha}_1 \right)$$

$$- b_0^+ \left(c_{-1} c_1 + \bar{c}_{-1} \bar{c}_1 \right) + \ldots$$

where we have dropped terms proportional to $L_0 - \bar{L}_0$ and b_0^- that annihilate any string field. Note that acting on the present string field the " -2 " on the first line cancels with the contributions from the second line. This gives

其中我们去掉了正比于 $L_0 - \bar{L}_0$ 和 b_0^- 、会湮灭任意弦场的项。注意作用在当前弦场上时，第一行的“-2”会和第二行的贡献抵消，由此得到

$$S^{(2)} = (\alpha')^{-D/2} \frac{1}{8 g_s^2} \int d^D x \left[\frac{\alpha'}{4} e^{\mu\nu} \Box e_{\mu\nu} + 2\alpha' \bar{e} \Box e - f^\mu f_\mu - \bar{f}^\mu \bar{f}_\mu \right.$$

$$\left. - \sqrt{\alpha'} f^\mu \left(\partial^\nu e_{\mu\nu} - 2 \partial_\mu \bar{e} \right) + \sqrt{\alpha'} \bar{f}^\nu \left(\partial^\mu e_{\mu\nu} + 2 \partial_\nu e \right) \right]. \tag{204}$$

The gauge parameter $|\Lambda\rangle$ for the linearized gauge transformations is

线性化规范变换的规范参数 $|\Lambda\rangle$ 为

$$|\Lambda\rangle = (\alpha')^{D/2} \frac{1}{g_s} \int \frac{d^D k}{(2\pi)^D} \left(\frac{i}{\sqrt{2}} \lambda_\mu(k) \alpha_{-1}^\mu c_1 - \frac{i}{\sqrt{2}} \overline{\lambda}_\mu(k) \overline{\alpha}_{-1}^\mu \bar{c}_1 + \mu(k) c_0^+ \right) |k\rangle.$$

(205)

The string field Λ has ghost number one and is annihilated by b_0^- . It encodes two vectorial gauge parameters λ_μ and $\overline{\lambda}_\mu$ and one scalar gauge parameter μ . The quadratic string action (202) is invariant under the gauge transformations

弦场 Λ 的鬼数为 1，且被 b_0^- 湮灭。它包含两个矢量规范参数 λ_μ 和 $\bar{\lambda}_\mu$，以及一个标量规范参数 μ。二次弦作用量 (202) 在如下规范变换下不变

$$\delta\,|\Psi\rangle = Q|\,\Lambda\rangle. \tag{206}$$

Expanding this equation gives the following gauge transformations of the component fields:

展开该式即可得到分量场的下列规范变换:

$$\delta e_{\mu\nu} = \sqrt{\alpha'}\left(\partial_\mu\bar{\lambda}_\nu + \partial_\nu\lambda_\mu\right)$$

$$\delta f_\mu = -\frac{1}{2}\alpha'\Box\lambda_\mu + \sqrt{\alpha'}\partial_\mu\mu,$$

$$\delta\bar{f}_\upsilon = \frac{1}{2}\alpha'\Box\bar{\lambda}_\upsilon + \sqrt{\alpha'}\partial_\upsilon\mu, \tag{207}$$

$$\delta e = -\frac{1}{2}\sqrt{\alpha'}\partial\cdot\lambda + \mu,$$

$$\delta\bar{e} = \frac{1}{2}\sqrt{\alpha'}\partial\cdot\bar{\lambda} + \mu.$$

This can be confirmed to be a symmetry of the action (204).

可以验证这是作用量 (204) 的对称性。

We can now introduce fields d and χ by

我们现在可以通过下式引入场 d 和 χ

$$d = \frac{1}{2}\left(e - \bar{e}\right), \text{ and } \chi = \frac{1}{2}\left(e + \bar{e}\right). \tag{208}$$

The gauge transformations of d and χ are

d 和 χ 的规范变换为

$$\delta d = -\frac{1}{4}\sqrt{\alpha'}\left(\partial\cdot\lambda + \partial\cdot\bar{\lambda}\right),$$

(209)

$$\delta\chi = -\frac{1}{4}\sqrt{\alpha'}\left(\partial\cdot\lambda - \partial\cdot\bar{\lambda}\right) + \mu.$$

We can use μ to make the gauge choice

我们可以利用 μ 选取如下规范

$$\chi = 0\,. \tag{210}$$

After this choice is made, gauge transformations with arbitrary λ and $\bar{\lambda}$ require compensating μ transformations to preserve $\chi = 0$. These do not affect d or $e_{\mu\upsilon}$ as neither transforms under μ gauge transformations. It does change the gauge transformations of f and $\bar{f}$, but this is of no concern here as these auxiliary fields will be eliminated using their equations of motion. Therefore, we set $e = d$ and $\bar{e} = -d$ in (204) and eliminate the auxiliary fields f_μ and $\bar{f}_\upsilon$. using their equations of motion:

选定该规范后，带任意参数 λ 和 $\bar{\lambda}$ 的规范变换需要引入补偿的 μ 变换来维持 $\chi = 0$ 。这不影响 d 或 $e_{\mu\upsilon}$ ，因为二者都不在 μ 规范变换下变换。这会改变 f 和 $\bar{f}$ 的规范变换，但我们在此无需关心这点，因为这些辅助场会通过它们的运动方程被消去。因此，我们令 (204) 中的 $e = d$ 和 $\bar{e} = -d$ 为零，并利用它们的运动方程消去辅助场 f_μ 和 $\bar{f}_\upsilon$:

$$f_\mu = -\frac{1}{2}\sqrt{\alpha'}\left(\partial^\upsilon e_{\mu\upsilon} - 2\partial_\mu \bar{e}\right),\ \bar{f}_\upsilon = \frac{1}{2}\sqrt{\alpha'}\left(\partial^\mu e_{\mu\upsilon} + 2\partial_\upsilon e\right). \tag{211}$$

The result is the following quadratic action:

最终得到如下二次作用量:

$$S^{(2)} = (\alpha')^{-(D-2)/2}\frac{1}{8g_s^2}\int d^D x\left[\frac{1}{4}e_{\mu\upsilon}\Box e^{\mu\upsilon} + \frac{1}{4}\left(\partial^\upsilon e_{\mu\upsilon}\right)^2 + \frac{1}{4}\left(\partial^\mu e_{\mu\upsilon}\right)^2\right.$$

$$\left. -2d\partial^\mu\partial^\nu e_{\mu\nu} - 4d\Box d\right]\,. \tag{212}$$

The gauge transformations are

规范变换为

$$\delta e_{\mu\nu} = \sqrt{\alpha'}\left(\partial_\nu \lambda_\mu + \partial_\mu \bar{\lambda}_\nu\right),$$

(213)

$$\delta d = -\frac{1}{4}\sqrt{\alpha'}\left(\partial\cdot\lambda + \partial\cdot\bar{\lambda}\right),$$

The action (212) and the associated gauge transformations are completely general.

作用量 (212) 和对应的规范变换具有完全的一般性。

With $e_{\mu\upsilon} = h_{\mu\upsilon} + b_{\mu\upsilon}$, the action (212) then gives

代入 $e_{\mu\upsilon} = h_{\mu\upsilon} + b_{\mu\upsilon}$ 后，作用量 (212) 给出

$$S^{(2)} = (\alpha')^{-(D-2)/2}\frac{1}{8g_s^2}\int d^D xL\left[h, b, d\right], \tag{214}$$

where

其中

$$L[h,b,d] = \frac{1}{4}h^{\mu\upsilon}\partial^2 h_{\mu\upsilon} + \frac{1}{2}\left(\partial^\upsilon h_{\mu\upsilon}\right)^2 - 2d\partial^\mu\partial^\upsilon h_{\mu\upsilon} - 4d\partial^2 d$$

$$+\frac{1}{4}b^{\mu\upsilon}\partial^2 b_{\mu\upsilon} + \frac{1}{2}\left(\partial^\upsilon b_{\mu\upsilon}\right)^2. \tag{215}$$

To appreciate this result, we recall the standard low-energy effective string action S_{st} for gravity, Kalb-Ramond, and dilaton fields:

为了说明这个结果的意义，我们回顾引力、Kalb-Ramond 场和 dilaton 场的标准弦低能有效作用量 S_{st}：

$$S_{\mathrm{st}} = \frac{1}{2\kappa^2}\int d^D x\sqrt{-\mathrm{g}}e^{-2\phi}\left[R - \frac{1}{12}H^2 + 4(\partial\phi)^2\right]. \tag{216}$$

We expand to quadratic order in fluctuations using

我们利用下式将涨落展开到二阶项

$$g_{\mu\upsilon} = \eta_{\mu\upsilon} + h_{\mu\upsilon},\ \phi = d + \frac{1}{4}\eta^{\mu\upsilon}h_{\mu\upsilon},\ H_{\mu\upsilon\rho} = \partial_\mu b_{\upsilon\rho} + \cdots, \tag{217}$$

and after a long but familiar calculation involving the expansion of R , we find

经过包含 R 展开的冗长但常规的计算后，我们得到

$$S_{\mathrm{st}}^{(2)} = \frac{1}{2\kappa^2}\int d^D x L[h,b,d], \tag{218}$$

the exact same result we had for the string field quadratic action, provided we make the identification

当我们做如下替换时，所得结果与弦场二次作用量的结果完全一致

$$\kappa = (\alpha')^{(D-2)/4}(2g_s). \tag{219}$$

This also relates the string field fluctuations to the fluctuations of the metric, Kalb-Ramond field, and dilaton.

这也将弦场涨落与度规、卡尔布-朗道场和 dilaton 的涨落联系起来。

We now turn to the symmetries. The linearized version of the standard action (218) is invariant under linearized diffeomorphisms, with parameter ε_μ and antisymmetric tensor gauge transformations with parameter $\tilde{\varepsilon}_\upsilon$:

现在我们来讨论对称性。标准作用量 (218) 的线性化形式在参数为 ε_μ 的线性化微分同胚变换，以及参数为 $\tilde{\varepsilon}_\upsilon$ 的反对称张量规范变换下不变:

$$\delta h_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$$

$$\delta b_{\mu\nu} = -\partial_\mu \tilde{\varepsilon}_\nu + \partial_\nu \tilde{\varepsilon}_\mu \tag{220}$$

$$\delta d = -\frac{1}{2} \partial \cdot \varepsilon$$

Note that the scalar dilaton $\phi \equiv d + \frac{1}{4}\eta^{\mu\upsilon} h_{\mu\upsilon}$ is invariant under linearized diffeomorphisms. The gauge symmetries (213) of the SFT action coincide with the familiar ones above if we let

注意标量 dilaton $\phi \equiv d + \frac{1}{4}\eta^{\mu\upsilon} h_{\mu\upsilon}$ 在线性化微分同胚变换下不变。若我们令 SFT 作用量的规范对称性 (213) 取如下形式，它就与上述我们熟知的规范对称性一致:

$$\varepsilon_\mu \equiv \frac{1}{2}\sqrt{\alpha'}\left(\lambda_\mu + \bar{\lambda}_\mu\right), \ \tilde{\varepsilon}_\mu \equiv \frac{1}{2}\sqrt{\alpha'}\left(\lambda_\mu - \bar{\lambda}_\mu\right). \tag{221}$$

This shows that the quadratic string field theory action has all the requisite properties. Part of this action involving the graviton and the dilaton field was obtained in [63]. A closely related action was derived in setting up double field theory [64].

这说明二次弦场论作用量具备所有要求的性质。该作用量中包含引力子和 dilaton 场的部分已在文献 [63] 中得到。一个密切相关的作用量是在建立双场论的过程中导出的 [64]。

Properties of String Field Theory

弦场论的性质

In this section, we shall describe some properties of string field theory developed in section ”Bosonic and Superstring Field Theories”. In this analysis, we assume that conformal field theory correlation functions follow the gluing axioms mentioned at the end of . We will discuss the propagators for the various theories and how, at least formally, the world-sheet amplitudes arise from string field theory. We then turn to the sign convention of forms, focusing on the case of open-closed string field theory. We continue with a discussion of the 1PI and Wilsonian effective action for string field theory and justify the equivalence of theories using different string vertices using field redefinitions. The property of background independence of string field theory is reviewed next, followed by the dilaton theorem.

本节我们将介绍在“玻色弦与超弦场论”一节中建立的弦场论的若干性质。在分析中，我们假设共形场论关联函数满足末尾提到的粘合公理。我们将讨论不同理论的传播子，以及世界面振幅至少在形式上是如何从弦场论中产生的。随后我们转向形式的符号约定，重点讨论开-闭弦场论的情况。我们接着讨论弦场论的 1PI(单粒子不可约) 和威尔逊有效作用量，并论证通过场重定义，使用不同弦顶点的理论是等价的。接下来我们回顾弦场论的背景无关性，之后介绍 dilaton 定理。

World-Sheet String Amplitudes from String Field Theory

弦场论导出世界面弦振幅

One of the consistency conditions that any version of string field theory needs to satisfy is that the perturbative amplitudes computed from string field theory should formally agree with the ones described in section ”Bosonic String Amplitudes and Their Off-Shell Generalization” in first quantized string field theory. Here we use the word ”formally” because the integration over moduli spaces of Riemann surfaces for the world-sheet amplitudes in section ”Bosonic String Amplitudes and Their Off-Shell Generalization” often diverges as we approach noded surfaces and must be regulated by using insights from string field theory. This will be discussed later in section ”Mass Renormalization and Vacuum Shift”. In this section, we shall discuss how the formal equivalence between string field theory amplitudes and the world-sheet amplitudes arises.

任何形式的弦场论都需要满足一个一致性条件: 从弦场论计算得到的微扰振幅，需要与第一量子化弦理论中“玻色弦振幅及其脱壳推广”一节描述的振幅形式一致。我们这里用“形式上”一词，是因为“玻色弦振幅及其脱壳推广”一节中对世界面振幅的黎曼模空间积分，在趋近结点曲面时通常会发散，必须借助弦场论的结论进行正则化，我们会在后面“质量重正规化与真空移位”一节对此进行讨论。本节我们将讨论弦场论振幅与世界面振幅之间的形式等价性是如何产生的。

We shall first discuss this formal equivalence in the context of closed bosonic string theory. Note that with the normalization convention in (10), S is the Lorentzian action since the quadratic terms of physical fields with momentum p are proportional to $-p^2$. We shall compute the amplitudes in the Euclidean theory in order to avoid factors of i . The Euclidean action is given by $S_E = -S$, with S evaluated using the Euclidean metric and normalization conditions like (10) retaining their form with momenta replaced by Euclidean momenta. The minus sign is needed because S_E should have terms proportional to p^2 with positive sign. Therefore, the path-integral weight factor is $e^{-S_E} = e^S$. We shall use this convention to derive the Feynman rules discussed below. As a result, the kinetic operator K will be the negative of the operator that appears in the quadratic term in the action. For example, for a scalar field, we have $S = \int \left(-\frac{1}{2}\phi K \phi\right)$ with K the kinetic operator.

我们首先在闭玻色弦理论的框架下讨论这种形式等价性。注意在 (10) 的归一化约定下，S 是洛伦兹作用量，因为动量为 p 的物理场的二次项正比于 $-p^2$ 。为了避免出现 i 因子，我们在欧氏理论中计算振幅。欧氏作用量由 $S_E = -S$ 给出，其中 S 是用欧氏度规计算的，(10) 这类归一化条件保留形式，仅将动量替换为欧氏动量。引入负号是因为 S_E 需要包含正比于 p^2 的正号项，因此路径积分权重因子为 $e^{-S_E} = e^S$ 。我们将采用这一约定推导下文讨论的费曼规则，因此动能算符 K 是作用量二次项中出现的算符的负值。例如对标量场，我们有 $S = \int \left(-\frac{1}{2}\phi K \phi\right)$，其中 K 为动能算符。

In order to compute the amplitudes via a path integral, we need to first fix a gauge. As discussed in section ”Batalin-Vilkovisky Formalism”, this corresponds to setting the anti-fields to zero after suitable symplectic transformation. It follows from the discussion below (118) that one such choice is to set to zero the coefficients of the basis states annihilated by c_0^+ . Since the rest of the basis states are annihilated by b_0^+ , the gauge condition can also be written as

要通过路径积分计算振幅，我们首先需要固定规范。正如“巴塔林-维尔可夫斯基形式”一节所讨论的，这对应于经过适当辛变换后将反场设为零。根据 (118) 下方的讨论，其中一种选择是将被 c_0^+ 湮灭的基态的系数设为零。由于其余基态都被 b_0^+ 湮灭，规范条件也可以写为

$$b_0^+|\Psi\rangle = 0. \tag{222}$$

This is known as the Siegel gauge. In this gauge, the action (114) takes the form

这就是著名的西格尔规范。在此规范下，作用量 (114) 形式为

$$S = \frac{1}{2}\langle\Psi|c_0^- c_0^+ L_0^+|\Psi\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\{\Psi^n\}. \tag{223}$$

The kinetic operator here is $K = -c_0^- c_0^+ L_0^+$. The bosonic string propagator $\mathcal{P}_b$ is obtained by inverting the kinetic operator term, $\mathcal{P}_b K = \mathbf{1}$, on the space of states annihilated by b_0^+ . This gives

此处的动能算符为 $K = -c_0^- c_0^+ L_0^+$ 。玻色弦传播子 $\mathcal{P}_b$ 可通过在被 b_0^+ 湮灭的态空间上对动能项 $\mathcal{P}_b K = \mathbf{1}$ 求逆得到，结果为

$$\mathcal{P}_b = -b_0^+ b_0^- \left(L_0 + \bar{L}_0\right)^{-1} \delta_{L_0,\bar{L}_0}. \tag{224}$$

The operator $\mathcal{P}_b$ is to be regarded as acting on the full CFT state space $\mathcal{H}'$; this is why we have put the last factor $\delta_{L_0,\bar{L}_0}$ to explicitly impose the constraint (99) that the string field satisfies. Since $L_0 - \bar{L}_0$ takes integer values in the full CFT state space, this can be written as

算符 $\mathcal{P}_b$ 定义在整个共形场论态空间 $\mathcal{H}'$ 上；因此我们引入最后一个因子 $\delta_{L_0,\bar{L}_0}$ 来显式施加弦场满足的约束 (99)。由于 $L_0 - \bar{L}_0$ 在整个共形场论态空间上取整数值，上式也可以写为

$$\begin{aligned}\mathcal{P}_b &= -\frac{1}{2\pi} b_0^+ b_0^- \int_0^{2\pi} d\theta \int_0^{\infty} ds e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} \\ &= \frac{1}{\pi} b_0 \bar{b}_0 \int_{|q|\le 1} \frac{d^2q}{|q|^2} q^{L_0} \bar{q}^{\bar{L}_0},\end{aligned} \tag{225}$$

where we have defined

其中我们定义

$$q \equiv e^{-s+i\theta},\ d^2q = d\theta ds|q|^2 \equiv \frac{i}{2} dq \wedge d\bar{q}, \tag{226}$$

The two-form d^2q is the exact analog of $d^2z \equiv dx \wedge dy$ for the real plane, with $z = x + iy$. It is the ”area” form giving a positive result acting on a basis with the standard orientation.

二形式 d^2q 是实平面上 $d^2z \equiv dx \wedge dy$ 的精确类比，满足 $z = x + iy$ 。它是对标准定向的基作用给出正结果的“面积”形式。

We can give the following geometric interpretation to the contribution from a Feynman diagram where a pair of interaction vertices is connected by a propagator. Associated with each interaction vertex, we have a chain $\mathcal{V}_{g_i,n_i}$ of $\widehat{\mathcal{P}}_{g_i,n_i}$ for $i = 1, 2$. Let us pick a particular point in $\mathcal{V}_{g_i,n_i}$ for each of the two vertices. This gives a particular Riemann surface with punctures and a choice of local coordinate (up to a phase) at each of the punctures. Now in the Feynman diagram, one of the punctures at each of the vertices is connected to the internal propagator. If we denote by w_1 and w_2 the local coordinates at these punctures, then one can show, using properties of CFT on Riemann surfaces, that for a given value of q in (225), the contribution from the Feynman diagram can be expressed as correlation function on a new Riemann surface of genus $g_1 + g_2$ and $n_1 + n_2 - 2$ punctures, obtained by gluing the original pair of Riemann surfaces using the relation [65, 66] .[9]

我们可以对一对相互作用顶点由传播子连接的费曼图的贡献给出如下几何诠释。对每个相互作用顶点，我们关联一条 $\mathcal{V}_{g_i,n_i}$ 链，对应 $i = 1, 2$ 的 $\widehat{\mathcal{P}}_{g_i,n_i}$ 。我们为两个顶点各自在 $\mathcal{V}_{g_i,n_i}$ 中选取一个特定点，这就得到一个带孔的特定黎曼曲面，并且在每个孔处选定了局部坐标 (相差一个相位)。现在在费曼图中，每个顶点各有一个孔连接到内部传播子。如果我们用 w_1 和 w_2 标记这些孔处的局部坐标，那么利用黎曼曲面上共形场论的性质可以证明: 对于 (225) 式中 q 的给定取值，该费曼图的贡献可以表示为一个亏格为 $g_1 + g_2$ 、带有 $n_1 + n_2 - 2$ 个孔的新黎曼曲面上的关联函数，这个新黎曼曲面是通过关系 [65, 66] .[9] 将原一对黎曼曲面粘合得到的

$$w_1 w_2 = q \tag{227}$$

If we now span the whole range $|q| \leq 1$ and also consider the collection of all points in the chains $\mathcal{V}_{g_i,n_i}$, we get a chain in $\widehat{\mathcal{P}}_{g_1+g_2,n_1+n_2-2}$ of dimension:

如果我们现在遍历整个 $|q| \leq 1$ 取值范围，同时考虑链 $\mathcal{V}_{g_i,n_i}$ 中所有点的集合，我们就得到 $\widehat{\mathcal{P}}_{g_1+g_2,n_1+n_2-2}$ 中一条维数为如下的链:

$$(6g_1 - 6 + 2n_1) + (6g_2 - 6 + 2n_2) + 2, \tag{228}$$

where the last additive contribution of 2 represents the two-dimensional space spanned by q . This has the correct dimension $6(g_1 + g_2) - 6 + 2(n_1 + n_2 - 2)$ that is needed to describe a string amplitude at genus $g_1 + g_2$ with $n_1 + n_2 - 2$ external legs. Since $dq \wedge d\bar{q} = -2id^2q$, the propagator can be written as

其中最后一项加性贡献 2 代表 q 张成的二维空间。这正好具有描述亏格为 $g_1 + g_2$ 、带有 $n_1 + n_2 - 2$ 条外腿的弦振幅所需的正确维数 $6(g_1 + g_2) - 6 + 2(n_1 + n_2 - 2)$ 。由于 $dq \wedge d\bar{q} = -2id^2q$ ，传播子可以写为

$$\mathcal{P}_b = \int_{|q|\leq 1} \left[\left(-\frac{1}{2\pi i} \right) b_0 \bar{b}_0 \frac{dq \wedge d\bar{q}}{|q|^2} \right] q^{L_0} \bar{q}^{\bar{L}_0}. \tag{229}$$

The term inside the square bracket defines the two-form $\Omega_{\mathcal{P}_b}$

方括号内的项定义了二形式 $\Omega_{\mathcal{P}_b}$

$$\Omega_{\mathcal{P}_b} = -\frac{1}{2\pi i} b_0 \bar{b}_0 \frac{dq \wedge d\bar{q}}{|q|^2}. \tag{230}$$

Let us compare this with the result we shall get if we had followed the procedure used in defining the canonical forms $\widehat{\Omega}$ given in (68), restricted to purely closed string amplitudes on surfaces without boundaries for which there is no sign ambiguity. For the above gluing $w_1 w_2 = q$, we have $w_1 = F(w_2, q)$ and the derivatives

我们将这个结果与遵循 (68) 式给出的正则形式 $\widehat{\Omega}$ 定义流程得到的结果做比较，该流程限制在无边界曲面上无符号歧义的纯闭弦振幅的情况。对于上述粘合操作 $w_1 w_2 = q$ ，我们有 $w_1 = F(w_2, q)$ 以及导数

$$\frac{\partial F}{\partial q} = \frac{1}{w_2} = \frac{w_1}{q}, \ \frac{\partial \bar{F}}{\partial q} = 0; \ \frac{\partial F}{\partial \bar{q}} = 0, \ \frac{\partial \bar{F}}{\partial \bar{q}} = \frac{\bar{w}_1}{\bar{q}}. \tag{231}$$

This means that the canonical form $\widehat{\Omega}_b$ arising from the gluing operation is

这说明粘合操作得到的正则形式 $\widehat{\Omega}_b$ 为

$$\begin{aligned} \widehat{\Omega}_b = \mathcal{B}\left[\frac{\partial}{\partial q}\right]\mathcal{B}\left[\frac{\partial}{\partial \bar{q}}\right] dq \wedge d\bar{q} &= \frac{1}{q}\oint w_1 b(w_1)\, dw_1 \times \frac{1}{\bar{q}}\oint \bar{w}_1 \bar{b}(\bar{w}_1)\, d\bar{w}_1 \times dq \wedge d\bar{q} \\ &= \frac{b_0}{q}\frac{\bar{b}_0}{\bar{q}} dq \wedge d\bar{q}. \end{aligned} \tag{232}$$

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[9] In doing this identification explicitly, we must choose the phase of the coordinates w_1 and w_2 . This can't be done globally in a continuous way, but no difficulty arises since the propagator actually integrates over all twist angles.

[9] 在显式进行这个识别时，我们必须选定坐标 w_1 和 w_2 的相位。这无法在全局以连续方式完成，但不会产生问题，因为传播子实际上会对所有扭转角积分。

We then see that the propagator two-form $\Omega_{\mathcal{P}_b}$ is simply related to the canonical form

我们之后可以看到，传播子二形式 $\Omega_{\mathcal{P}_b}$ 与正则形式有简单的关系

$$\Omega_{\mathcal{P}_b} = -\frac{1}{2\pi i}\widehat{\Omega}_b. \tag{233}$$

The numerical factor is the normalization factor of the propagator form. We now confirm the consistency of the normalization factors. When we use the propagator to glue a form on $\widehat{\mathcal{P}}_{g_1,n_1}$ to a form in $\widehat{\mathcal{P}}_{g_2,n_2}$ the normalization factors must give the normalization factor of $\mathcal{P}_{g_1+g_2,n_1+n_2-2}$. They do indeed:

数值因子是传播子形式的归一化因子。我们现在验证归一化因子的自洽性: 当我们用传播子将 $\widehat{\mathcal{P}}_{g_1,n_1}$ 上的一个形式与 $\widehat{\mathcal{P}}_{g_2,n_2}$ 中的一个形式粘合时，归一化因子的乘积必须给出 $\mathcal{P}_{g_1+g_2,n_1+n_2-2}$ 的归一化因子，事实确实如此:

$$\left(-\frac{1}{2\pi i}\right)^{3g_1-3+n_1}\left(-\frac{1}{2\pi i}\right)^{3g_2-3+n_2}\left(-\frac{1}{2\pi i}\right)^{1}=\left(-\frac{1}{2\pi i}\right)^{3(g_1+g_2)-3+(n_1+n_2-2)},$$

(234)

where we used (44). This is in fact the origin of the normalization factor in (44). The normalization factors also work out when the propagator connects two lines from the same vertex.

其中我们用到了 (44) 式。这实际上就是 (44) 式中归一化因子的来源。当传播子连接同一顶点的两条线时，归一化因子也同样自洽。

By repeated use of the propagator, one can show that each Feynman diagram contributing to an n -point amplitude at order g_s^{2g-2+n} gives an integral of the form (41) that runs over a chain. For example, the elementary g -loop, n -point vertex, gives integration over $\mathcal{V}_{g,n}$. The geometric BV master equation (104) guarantees that the boundaries of these chains fit together so that the sum over all Feynman diagrams gives integration over the full chain $\mathcal{F}_{g,n}$. This reproduces the amplitude (41) given in the world-sheet formalism.

通过重复使用传播子，可以证明，对 n 点振幅有贡献的每一阶 g_s^{2g-2+n} 费曼图，都会给出形如 (41) 的、在一个链上的积分。例如，基本的 g 圈、n 点顶点对应在 $\mathcal{V}_{g,n}$ 上的积分。几何 BV 主方程 (104) 保证这些链的边界能够正确拼接，因此对所有费曼图求和后，就得到了在完整链 $\mathcal{F}_{g,n}$ 上的积分。这就重现了世界面形式下的振幅 (41)。

The analysis for type II or heterotic string theory is similar. In the Siegel gauge, $b_0^+|\Psi\rangle=0, b_0^+|\widetilde{\Psi}\rangle=0$, the action (127) takes the form

II 型弦论或杂化弦论的分析是类似的。在西格尔规范 $b_0^+|\Psi\rangle=0, b_0^+|\widetilde{\Psi}\rangle=0$ 下，作用量 (127) 形如

$$S=-\frac{1}{2}\langle\widetilde{\Psi}|c_0^-c_0^+L_0^+\mathcal{G}|\widetilde{\Psi}\rangle+\langle\widetilde{\Psi}|c_0^-c_0^+L_0^+|\Psi\rangle+\sum_{n=1}^{\infty}\frac{1}{n!}\{\Psi^n\}. \tag{235}$$

Therefore, the kinetic operator K_0 in $(\widetilde{\Psi},\Psi)$ space takes the form

因此，动能算符 K_0 在 $(\widetilde{\Psi},\Psi)$ 空间中的形式为

$$K_0=c_0^-c_0^+L_0^+\begin{pmatrix}\mathcal{G} & -1\\ -1 & 0\end{pmatrix}, \tag{236}$$

and the superstring propagator $\mathcal{P}_s$, given by the inverse of K_0 , is

而超弦传播子 $\mathcal{P}_s$ 由 K_0 的逆给出，即

$$\mathcal{P}_s=-b_0^+b_0^-\left(L_0+\bar{L}_0\right)^{-1}\delta_{L_0,\bar{L}_0}\begin{pmatrix}0 & 1\\ 1 & \mathcal{G}\end{pmatrix}. \tag{237}$$

Since $\widetilde{\Psi}$ does not appear in the interaction vertex, the only internal propagator needed for the computation of the Feynman diagrams is the Ψ−Ψ propagator. One can now analyze the Feynman diagrams as in the case of bosonic string field theory. The only difference is that the propagator contains the factor of $\mathcal{G}$. As a result, for a Feynman diagram with two elementary vertices joined by a propagator, the associated form in $\widehat{\mathcal{P}}_{g_1+g_2,n_1+n_2-2}$

has PCO insertions $\mathcal{X}_0$ and/or $\bar{\mathcal{X}}_0$ for R sector propagators. The boundary of this chain still coincides with part of the boundary of $\mathcal{V}_{g_1+g_2,n_1+n_2-2}$ since, as discussed above (124), the boundary of $\mathcal{V}_{g,n}$ has exactly the same PCO insertions. The rest of the analysis proceeds as before, and one can show that the sum of all the Feynman diagrams gives the integral of $\Omega^{(g,n)}_{6g-6+2n}$ over a chain $\widetilde{\mathcal{G}}_{g,n}$ of $\widehat{\mathcal{P}}^s_{g,n}$ that when pushed to $\mathcal{M}_{g,n}$ represents its fundamental homology class, thus reproducing the amplitude (41) given in the world-sheet formalism.

由于 $\widetilde{\Psi}$ 不出现在相互作用顶点中，计算费曼图只需要用到 $\Psi-\Psi$ 传播子这一种内传播子。接下来就可以像玻色弦场论的情况一样分析费曼图，唯一区别是传播子包含因子 $\mathcal{G}$ 。因此，对于两个基本顶点由一个传播子连接的费曼图，其在 $\widehat{\mathcal{P}}_{g_1+g_2,n_1+n_2-2}$ 中对应的形式对 R 区传播子会有 PCO 插入 $\mathcal{X}_0$ 和/或 $\bar{\mathcal{X}}_0$ 。该链的边界仍然与 $\mathcal{V}_{g_1+g_2,n_1+n_2-2}$ 的部分边界重合，因为正如上文 (124) 所讨论，$\mathcal{V}_{g,n}$ 的边界恰好有相同的 PCO 插入。其余分析和之前一样，可以证明所有费曼图的和给出 $\Omega^{(g,n)}_{6g-6+2n}$ 在 $\widehat{\mathcal{P}}^s_{g,n}$ 的链 $\widetilde{\mathcal{G}}_{g,n}$ 上的积分，将该链推入 $\mathcal{M}_{g,n}$ 后就对应其基本同调类，从而重现世界面形式下的振幅 (41)。

The computation of amplitudes in theories that include open strings also includes Feynman diagrams with open string propagators. In the Siegel gauge $b_0\,|\Psi_o\rangle=0$, the kinetic operator is $K=-c_0L_0$, and therefore, open string propagator $\mathcal{P}_o$, satisfying $\mathcal{P}_oK=\mathbf{1}$ on the space of states annihilated by b_0 , is given by

含开弦的理论中计算振幅时，还会包含带开弦传播子的费曼图。在西格尔规范 $b_0\,|\Psi_o\rangle==0$ 下，动能算符为 $K=-c_0L_0$ ，因此在被 b_0 零化的态空间中满足 $\mathcal{P}_oK=\mathbf{1}$ 的开弦传播子 $\mathcal{P}_o$ 为

$$\mathcal{P}_o=-b_0L_0^{-1}=-b_0\int_0^1\frac{dq_o}{q_o}q_o^{L_0}=\int_0^1\left[(-b_0)\,\frac{dq_o}{q_o}\right]q_o^{L_0}. \tag{238}$$

Here q_o is a real integration variable. The term inside the square bracket defines the one-form $\Omega_{\mathcal{P}_o}$

这里 q_o 是实积分变量。方括号内的项定义了一元形式 $\Omega_{\mathcal{P}_o}$

$$\Omega_{\mathcal{P}_o}=-b_0\frac{dq_o}{q_o}. \tag{239}$$

Geometrically, when we connect two open string punctures by an open string propagator, the local coordinates w_1 and w_2 around the open string punctures get identified via

从几何上看，当我们用开弦传播子连接两个开弦 puncture 时，开弦 puncture 周围的局部坐标 w_1 和 w_2 通过下式等同:

$$w_1w_2=-q_o,\ q_o\in[0,1]\,. \tag{240}$$

We can now identify w_1 as σ_s and w_2 as τ_s in the language of section "Bosonic String Amplitudes and Their Off-Shell Generalization". Then we have $w_1=\ G(w_2,q_o)$, with $\frac{\partial G}{\partial q_o}=w_1/q_o$. The canonical form in this open string case is

现在我们可以借助“玻色弦振幅及其离壳推广”一节的语言，将 w_1 识别为 σ_s ，将 w_2 识别为 τ_s 。由此我们得到 $w_1=\ G(w_2,q_o)$ ，其中 $\frac{\partial G}{\partial q_o}=w_1/q_o$ 。该开弦情形下的标准型为

$$\widehat{\Omega}_o = \mathcal{B}\left[\frac{\partial}{\partial q_o}\right] dq_o \tag{241}$$

where

其中

$$\mathcal{B}\left[\frac{\partial}{\partial q_o}\right] = \int dw_1 b(w_1) \frac{w_1}{q_o} + \int d\bar{w}_1 \bar{b}(\bar{w}_1) \frac{\bar{w}_1}{q_o} = \frac{1}{q_o} \oint_C dw_1 b(w_1) w_1,$$

(242)

where in the last step, we have used the doubling trick. Now according to the prescription given in section ”Bosonic String Amplitudes and Their Off-Shell Generalization”, on the glued Riemann surface, the integration contour C must keep the region covered by the w_1 coordinate system to the left. If we take the gluing to be on the line $|w_1| = |w_2| = q_o^{1/2}$, then the w_1 coordinate system will cover the region $|w_1| \geq q_o^{1/2}$. This leads to C being a clockwise contour around $w_1 = 0$, as indicated by the $\oint$ symbol. We can now identify $\oint_C dw_1 b(w_1) w_1$ as $-b_0$ and get

其中我们在最后一步使用了加倍技巧。现在根据“玻色弦振幅及其离壳推广”一节给出的规则, 在粘合后的黎曼曲面上，积分围道 C 必须让 w_1 坐标系覆盖的区域保持在左侧。如果我们沿直线 $|w_1| = |w_2| = q_o^{1/2}$ 进行粘合，那么 w_1 坐标系将覆盖区域 $|w_1| \geq q_o^{1/2}$ 。这使得 C 成为绕 $w_1 = 0$ 的顺时针围道，正如 $\oint$ 符号所标示的。我们现在可以将 $\oint_C dw_1 b(w_1) w_1$ 识别为 $-b_0$ ，得到

$$\mathcal{B}\left[\frac{\partial}{\partial q_o}\right] = -\frac{1}{q_o} b_0, \ \widehat{\Omega}_o = -b_0 \frac{dq_o}{q_o}. \tag{243}$$

Comparing this with (239), we get the relation:

与 (239) 对比，我们得到关系:

$$\Omega_{\mathcal{P}_o} = \widehat{\Omega}_o \tag{244}$$

When we connect two vertices by an open string propagator, then the corresponding contribution to the amplitude is obtained by gluing the $\mathcal{V}_{g,b,n_c,n_o}$ corresponding to the vertices by the gluing operation (240) picking one open string puncture from each vertex. Using this, we can carry out the analog of the computation (234) for the open string propagator. Consider the effect of joining a pair of surfaces of type $(g_1, b_1, n_{c1}, n_{o1})$ and $(g_2, b_2, n_{c2}, n_{o2})$ via an open string propagator to construct a surface of type $(g_1 + g_2, b_1 + b_2 - 1, n_{c1} + n_{c2}, n_{o1} + n_{o2} - 2)$. Equation (244) tells us that we must have

当我们用一个开弦传播子连接两个顶点时，振幅对应的贡献可以通过如下方式得到: 从每个顶点选取一个开弦 puncture，利用粘合操作 (240) 将顶点对应的 $\mathcal{V}_{g,b,n_c,n_o}$ 粘合起来。利用这一点，我们可以对开弦传播子做类似 (234) 的计算。考虑通过开弦传播子连接一张 $(g_1, b_1, n_{c1}, n_{o1})$ 型曲面和一张 $(g_2, b_2, n_{c2}, n_{o2})$ 型曲面，构造一张 $(g_1 + g_2, b_1 + b_2 - 1, n_{c1} + n_{c2}, n_{o1} + n_{o2} - 2)$ 型曲面的效应。根据 (244) 式，我们必然有

$$N_{g_1,b_1,n_{c1},n_{o1}} N_{g_2,b_2,n_{c2},n_{o2}} \sim N_{g_1+g_2,b_1+b_2-1,n_{c1}+n_{c2},n_{o1}+n_{o2}-2}, \tag{245}$$

where $\sim$ denotes equality up to a sign arising from rearrangement of the ghost insertions and vertex operators inside the correlation function and possible mismatch between the orientations in the integration measure over the moduli space. Our N ' s in (84) indeed satisfy this. Similar tests can be performed when the open string propagator connects two punctures on the same surface, either on the same boundary or on two different boundaries, leading to the conditions

其中 $\sim$ 表示等式仅相差一个符号，该符号来源于关联函数内鬼插入和顶点算子的重排，以及模空间积分测度中可能存在的取向不匹配。我们 (84) 式中的 N 确实满足这一点。当开弦传播子连接同一曲面上的两个 puncture(无论是在同一边界还是两个不同边界上) 时，也可以做类似检验，由此得到条件

$$N_{g,b,n_c,n_o} \sim N_{g,b+1,n_c,n_o+2},\ N_{g,b,n_c,n_o} \sim N_{g+1,b-1,n_c,n_o+2}. \tag{246}$$

In open closed string field theory, similar results associated with closed string propagators will also have sign ambiguities, since even though the ghost insertions associated with the propagator is Grassmann even, after gluing, we may have to rearrange the ghost insertions and vertex operators on the original Riemann surface(s) to bring them to the desired arrangement on the final Riemann surface. This leads to the relations

在开-闭弦场论中，与闭弦传播子相关的类似结果也同样存在符号不确定性，这是因为即使传播子对应的鬼插入是格拉斯曼偶的，粘合后我们仍可能需要重排原黎曼曲面上的鬼插入和顶点算子，才能得到最终黎曼曲面上所需的排布。由此得到关系

$$\eta_c N_{g_1,b_1,n_{c1},n_{o1}} N_{g_2,b_2,n_{c2},n_{o2}} \sim N_{g_1+g_2,b_1+b_2,n_{c1}+n_{c2}-2,n_{o1}+n_{o2}},$$

(247)

$$\eta_c N_{g-1,b,n_c+2,n_o} \sim N_{g,b,n_c,n_o}.$$

In section "Signs of Forms in $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ ", we have described the prescription for the signs of $\widehat{\Omega}$ for which (245),(246), and (247) hold exactly including signs, with $\sim$ replaced by $=$, and as a result (84) also hold exactly. For a proof of this statement, we ask the readers to consult [22]. This analysis determines the normalization of $\Omega_p^{(g,b,n_c,n_o)}$ when the degree p is equal to the dimensionality of $\mathcal{M}_{g,b,n_c,n_o}$, since these are the forms that appear directly in the expression for amplitudes. Indirectly, however, it determines the normalization of forms of different degrees using (52), suitably generalized to include open strings.

在“$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 中形式的符号”一节中，我们已经描述了 $\widehat{\Omega}$ 的符号规则，在该规则下，(245)、(246) 和 (247)(包括符号在内) 完全成立，只需将 $\sim$ 替换为 $=$ ，相应地 (84) 也完全成立。关于该结论的证明，请读者参考文献 [22]。当次数 p 等于 $\mathcal{M}_{g,b,n_c,n_o}$ 的维数时，该分析确定了 $\Omega_p^{(g,b,n_c,n_o)}$ 的正规化，因为这类形式会直接出现在振幅的表达式中。而通过推广到开弦的 (52) 式，它也能间接确定不同次数形式的正规化。

With the Feynman rules derived above, the tree amplitudes of tree-level open string field theory described in section "Tree-Level Open String Field Theory" reproduce the disk amplitudes of the world-sheet theory with external open strings. On the other hand, the full set of Feynman diagrams in the open-closed string field theory discussed in section "Open-Closed String Field Theory", which now contain both open and closed string propagators, reproduce the full amplitudes given by the world-sheet theory of open and closed strings.

利用上文推导出的费曼规则，“树级开弦场论”一节中描述的树级开弦场论树振幅，恰好重现了带外开弦的世界面理论的圆盘振幅。另一方面，“开-闭弦场论”一节中讨论的开-闭弦场论的完整费曼图同时包含开弦和闭弦传播子，恰好重现了开弦闭弦世界面理论给出的完整振幅。

If we compute loop amplitudes with the vertices and propagators of tree-level open string field theory, we find that as in the case of closed string theory, the Feynman diagrams constructed from tree level vertices and propagator cover only a part of the moduli space, and we need to add new contributions to fill the missing regions. However, unlike in the case of tree-level closed string field theory, where all higher-genus vertices contain only regular Riemann surfaces, in the case of tree-level open string field theory, the missing regions include nodal Riemann surfaces associated with closed string degeneration. To avoid singular vertices, we must also add the closed strings to the theory and consider a field theory of open and closed strings, in which these singular regions can arise from Feynman diagrams with closed string propagators. The only exception to this is the bosonic cubic open SFT where the loop amplitudes constructed from the tree-level Feynman rules reproduce all the loop amplitudes [67, 68] .

如果我们用树级开弦场论的顶点和传播子计算圈振幅，会发现和闭弦场论的情况一样，由树级顶点和传播子构造的费曼图仅覆盖模空间的一部分，我们需要添加新的贡献来填补缺失区域。但和树级闭弦场论不同——树级闭弦场论中所有高亏格顶点仅包含正则黎曼曲面，而树级开弦场论的缺失区域包含与闭弦退化相关的结点黎曼曲面。为了避免奇异顶点，我们必须在理论中引入闭弦，考虑开弦和闭弦的场论，此时这些奇异区域可以由带闭弦传播子的费曼图产生。唯一的例外是玻色立方开弦场论，该理论中由树级费曼规则构造的圈振幅可以重现所有圈振幅 [67, 68] 。

Results for Special Amplitudes

特殊振幅的结果

In this subsection, we shall derive some specific results for specific string amplitudes. This includes analysis of the disk one-point function of closed strings to find the relation between the constant K and the D-brane tension $\mathcal{T}$, the effect of insertion of a closed string or an open string to a given string amplitude, the disk two-point function of one open and one closed string, and the disk two-point function of two closed strings.

在本小节中，我们将推导特殊弦振幅的一些具体结果，包括: 分析闭弦的圆盘单点函数以得到常数 K 与 D 膜张力 $\mathcal{T}$ 之间的关系、往给定弦振幅中插入闭弦或开弦的效应、一个开弦一个闭弦的圆盘两点函数，以及两个闭弦的圆盘两点函数。

We shall first find the relation between the constant K appearing in (12) and the tension $\mathcal{T}$ of the corresponding D-brane. For this, we consider a string field of the form (196)

我们首先求出式 (12) 中出现的常数 K 和对应 D 膜的张力 $\mathcal{T}$ 之间的关系。为此，我们考虑形如 (196) 的弦场

$$\mathcal{O}_h = \frac{1}{g_s} h_{\mu\nu} c\bar{c} \partial X^\mu \bar{\partial} X^\nu, \tag{248}$$

with constant $h_{\mu\upsilon}$, with μ, υ , directions along the world-volume of the D-brane, and setting $\alpha' = 1$. Comparison with the string field (197) shows that this corresponds to setting $e_{\mu\upsilon}$ to $h_{\mu\upsilon}$ and $e, \bar{e}, f_\mu, \bar{f}_\mu$ equal to zero. From (208), we now see that this corresponds to setting the string dilaton d to zero, and hence from (217), we get

其中 $h_{\mu\upsilon}$ 为常数，μ, υ 是沿 D 膜世界体的方向，且取 $\alpha' = 1$ 。与弦场 (197) 对比可知，这对应将 $e_{\mu\upsilon}$ 取为 $h_{\mu\upsilon}$ ，$e, \bar{e}, f_\mu, \bar{f}_\mu$ 等于零。由式 (208) 可知这对应将弦 dilaton d 取为零，因此从式 (217) 我们得到

$$g_{\mu\upsilon} = \eta_{\mu\upsilon} + h_{\mu\upsilon}, \ \phi = \frac{1}{4}\eta^{\mu\upsilon}h_{\mu\upsilon}. \tag{249}$$

On the other hand, the action S_p of a Dp -brane in this background is given by

另一方面，该背景下 Dp 膜的作用量 S_p 由下式给出

$$S_p = -\mathcal{T}\int d^{p+1}x e^{-\phi}\sqrt{-g}. \tag{250}$$

With the help of (249), the terms linear in $h_{\mu\nu}$ in this expression can be written as

借助式 (249)，该表达式中 $h_{\mu\nu}$ 的一次项可以写为

$$S_p\big|_h = -\mathcal{T}\left\{\frac{1}{2}\eta^{\mu\upsilon}h^{\parallel}_{\mu\upsilon}\left(x_\perp = 0\right) - \phi\left(x_\perp = 0\right)\right\}$$

$$\int d^{p+1}x = -\frac{\mathcal{T}}{4}\eta^{\mu\upsilon}\left(h^{\parallel}_{\mu\upsilon} - h^{\perp}_{\mu\upsilon}\right)(2\pi)^{p+1}\delta^{(p+1)}(0), \tag{251}$$

where $h^{\parallel}_{\mu\nu}$ and $h^{\perp}_{\mu\nu}$ denote the component of $h_{\mu\nu}$ along the D-brane and transverse to the D-brane, respectively. This should be compared to the term in the open-closed string field theory linear in the string field due to the disk one-point function of closed strings. Using (160) and (84), we see that one-point disk amplitude of a closed string state A is given by

其中 $h^{\parallel}_{\mu\nu}$ 和 $h^{\perp}_{\mu\nu}$ 分别表示 $h_{\mu\nu}$ 沿 D 膜方向和垂直 D 膜方向的分量。我们需要将其与开弦闭弦场论中，由闭弦圆盘单点函数产生的弦场一次项对比。利用式 (160) 和 (84) 可得，闭弦态 A 的单点圆盘振幅为

$$\Omega_0^{(0,1,1,0)}(A) = -\frac{1}{2\pi i}\eta_c^{-1/2}\left\langle c_0^- A\right\rangle = \eta_c^{1/2}\left\langle c_0^- A\right\rangle. \tag{252}$$

For the closed string state given in (248), this becomes

对于式 (248) 给出的闭弦态，上式变为

$$\Omega_0^{(0,1,1,0)}(\mathcal{O}_h) = \eta_c^{1/2}\frac{1}{g_s}h_{\mu\upsilon}\left\langle c_0^- c\bar{c}\partial X^\mu\bar{\partial}X^\upsilon\right\rangle. \tag{253}$$

We can evaluate the correlator using the operator product expansion (7), the normalization condition (12), and the fact that for Neumann (Dirichlet) boundary condition, $\bar{X}^\mu$ can be replaced by ∂X^μ $(-\partial X^\mu)$ at the complex conjugate point. This gives the result for the term in the string field theory action

我们可以利用算子乘积展开 (7)、归一化条件 (12)，以及诺依曼 (狄利克雷) 边界条件下，$\bar{X}^\mu$ 在复共轭点可替换为 ∂X^μ $(-\partial X^\mu)$ 这一性质来计算关联函数，最终得到弦场论作用量中对应项的结果为

$$\Omega_0^{(0,1,1,0)}\left(\mathcal{O}_h\right)=\eta_c^{1/2}\frac{1}{g_s}\left(h_{\mu\nu}^{\parallel}-h_{\mu\nu}^{\perp}\right)\frac{K}{2}\eta^{\mu\nu}(2\pi)^{p+1}\delta^{(p+1)}(0)\,. \tag{254}$$

Comparing (251) with (254), we get

将式 (251) 与 (254) 对比，我们得到

$$K=-g_s\frac{\mathcal{T}}{2\sqrt{\eta c}}. \tag{255}$$

Note that K is a complex number. We can now use (255) and (174) to get

注意 K 是一个复数。现在我们可以利用式 (255) 和 (174) 得到

$$g_o^2=-\frac{2}{\mathcal{T}}\eta_c^2=\frac{1}{2\pi^2\mathcal{T}}\;\rightarrow\;\mathcal{T}g_o^2=\frac{1}{2\pi^2}, \tag{256}$$

as we quoted in (140).

即我们在式 (140) 中引用的结果。

Some specific application of these results have been discussed in [22]. Here we just summarize them, referring to [22] for the details.

这些结果的一些具体应用已经在文献 [22] 中讨论过。此处我们仅做总结，细节请参阅文献 [22]。

1. Let us suppose that we have an amplitude with fixed set of external states, given by an appropriate integral over the moduli space of an appropriate Riemann surface. If we want to insert another closed string state $\Psi_c = c\bar{c}V_c$ into the amplitude, then the net effect of this is to insert into the world-sheet correlation function a factor

1. 假设我们已有一个由合适黎曼曲面模空间上适当积分给出的、固定外态集合的振幅。如果我们要往振幅中插入另一个闭弦态 $\Psi_c = c\bar{c}V_c$ ，其整体效应是往世界面关联函数中插入一个因子

$$-\frac{g_s}{\pi}\int dy_R dy_I V_c(y)\,, \tag{257}$$

where we used $y = y_R + iy_I$.

此处我们用到了 $y = y_R + iy_I$ 。

2. We can get a similar result for open strings. For every insertion of an open string state $\psi_o = cV_o$ or $\Psi_o = K^{-1/2}cV_o$ into an amplitude for a dimension one matter primary V_o , we insert into the world-sheet correlator a factor of

2. 我们可以对开弦得到类似结果。每往一维物质主元 V_o 的振幅中插入一个开弦态 $\psi_o = cV_o$ 或 $\Psi_o = K^{-1/2}cV_o$，我们就要往世界面关联函数中插入一个因子

$$g_o \int dx V_o(x). \tag{258}$$

3. The disk one-point function of a closed string state with vertex operator $c\bar{c}V_c$ is given by

3. 带顶点算符 $c\bar{c}V_c$ 的闭弦态的圆盘单点函数由下式给出

$$-g_s \frac{\mathcal{T}}{2} \langle c_0^- c\bar{c}V_c \rangle'. \tag{259}$$

4. The disk two-point function of a closed string of vertex operator $c\bar{c}V_c$ and $\Psi_o = K^{-1/2}cV_o, \psi_o = cV_o$, is given by

4. 分别带顶点算符 $c\bar{c}V_c$ 和 $\Psi_o = K^{-1/2}cV_o, \psi_o = cV_o$ 的闭弦的圆盘两点函数由下式给出

$$-i\pi\mathcal{T} g_s g_o \langle c\bar{c}V_c cV_o \rangle'. \tag{260}$$

5. The disk two-point function of a pair of closed string states with vertex operators $\Psi_c^{(1)} = c\bar{c}V_c^{(1)}$ and $\Psi_c^{(2)} = c\bar{c}V_c^{(2)}$ is given by

5. 一对带顶点算符 $\Psi_c^{(1)} = c\bar{c}V_c^{(1)}$ 和 $\Psi_c^{(2)} = c\bar{c}V_c^{(2)}$ 的闭弦态的圆盘两点函数由下式给出

$$\frac{i}{2} g_s^2 \mathcal{T} \int dy \left\langle c\bar{c}V_c^{(1)}(i)(c+\bar{c})V_c^{(2)}(iy) \right\rangle'. \tag{261}$$

It is clear that some of the signs mentioned above can be changed by field redefinition. For example, if we redefine Ψ_c as $-\Psi_c$, it will lead to an extra factor of $(-1)^{n_c}$ in the amplitude and change the minus sign to plus sign on the right-hand sides of (257),(259), and (260). Similarly, by replacing Ψ_o by $-\Psi_o$, we can get an extra factor of $(-1)^{n_o}$ in the amplitude. However, once a few signs have been chosen this way, the signs of all other amplitudes get fixed according to the analysis given above.

显然，上述部分符号可通过场重新定义改变。例如，若我们将 Ψ_c 重新定义为 $-\Psi_c$，会在振幅中引入一个额外因子 $(-1)^{n_c}$，并将式 (257)、(259) 和 (260) 右侧的负号变为正号。同理，将 Ψ_o 替换为 $-\Psi_o$，我们可以在振幅中得到一个额外因子 $(-1)^{n_o}$。但一旦通过这种方式选定了少数符号，所有其他振幅的符号就会按照上述分析固定下来。

1PI Effective Action

1PI 有效作用量

In ordinary quantum field theories, the one-particle irreducible (1PI) effective action plays a very useful role. Operationally, the interaction vertices of the 1PI effective action are given by the sum of all Feynman diagrams contributing to an amputated Green's function, with the restriction that it should not be possible to divide the diagram into two disconnected parts by cutting a single internal propagator. Once such an action is constructed, the full Green's functions are obtained by computing tree diagrams of the 1PI effective field theory. Furthermore, the quantum corrected vacuum is given by a suitable extremum of the 1PI effective action, and the full quantum corrected propagator is given by the tree-level propagator computed from the quadratic part of the 1PI effective action. Therefore, the masses of particles can be read out directly from the zeroes of the kinetic operator of the 1PI effective action. Note that for a quantum field theory that has gauge symmetry, one must do gauge fixing to compute the Feynman diagrams required for the 1PI action. At the end, one may write the result for the action as a sum of a gauge-invariant effective action and a gauge-fixing term.

在普通量子场论中，单粒子不可约 (1PI) 有效作用量发挥着非常重要的作用。从操作层面来说，1PI 有效作用量的相互作用顶点由所有对截断格林函数有贡献的费曼图求和得到，且约束条件为: 无法通过切断单个内传播子将图分成两个不相连的部分。构建出这类作用量后，full 格林函数可通过计算 1PI 有效场论的树图得到。此外，量子修正真空由 1PI 有效作用量的合适极值给出，而完整的量子修正传播子则由 1PI 有效作用量二次项部分计算得到的树图级传播子给出。因此，粒子质量可以直接从 1PI 有效作用量动能算符的零点读出。注意，对于具有规范对称性的量子场论，必须先做规范固定才能计算 1PI 作用量所需的费曼图。最终，我们可以将作用量的结果写为规范不变有效作用量与规范固定项之和。

The 1PI effective action in string field theory can be defined in the same way. Remarkably, using the results of section "World-Sheet String Amplitudes from String Field Theory", it can also be constructed without explicit reference to gauge fixing, even though we implicitly use the Siegel gauge propagator. The clue is that for a string amplitude for n external closed string states at genus g , one can produce a space $\mathcal{F}_{g,n}$ in $\widehat{\mathcal{P}}_{g,n}$ that "covers" the moduli space $\mathcal{M}_{g,n}$ (in the sense that the projection forgetting the coordinates is a map of degree one) by adding to $\mathcal{V}_{g,n}$ the contributions of gluing lower-order vertices with operations of the form

弦场论中的 1PI 有效作用量可以用相同方式定义。值得注意的是，利用章节“弦场论导出的世界面弦振幅”的结果，它也可以无需明确提及规范固定来构造，即使我们隐含使用了西格尔规范传播子。关键在于，对于亏格 g 上 n 个外闭弦态的弦振幅，我们可以通过向 $\mathcal{V}_{g,n}$ 添加形如以下的操作粘合低阶顶点，构造出 $\widehat{\mathcal{P}}_{g,n}$ 中“覆盖”模空间 $\mathcal{M}_{g,n}$ 的空间 $\mathcal{F}_{g,n}$ (覆盖的含义是，忘掉坐标的投影是一次映射)

$$w_1 w_2 = q, \; |q| \leq 1, \tag{262}$$

with w_1, w_2 coordinates at the punctures of the surfaces (or surface) to be glued. These gluing operations follow the diagramatics of Feynman diagrams. Since q belongs to the full unit disk, this "disk gluing" creates surfaces, including degenerations arising for $q \to 0$ and corresponding to special "divisors" in the moduli space (see section "Geometric BV Master Equation and String Field Theory Master Equation"). To define the 1PI vertex, $\mathcal{V}_{g,n}^{\text{1PI}}$, one adds to the vertex $\mathcal{V}_{g,n}$ the surfaces from diagrams with disk-gluing operations, as long as the cutting of any gluing line does not split the Riemann surface into two pieces.

待粘合曲面上的 puncture 带有 w_1, w_2 个坐标。这些粘合操作遵循费曼图的绘图规则。由于 q 属于完整单位圆盘，这种“圆盘粘合”会生成新曲面，其中包含 $q \to 0$ 产生的退化，对应模空间中的特殊“除子”(参见“几何 BV 主方程与弦场论主方程”一节)。为定义 1PI 顶点 $\mathcal{V}_{g,n}^{1\mathrm{PI}}$，只要切割任意粘合线都不会将黎曼曲面拆分为两部分，我们就会将带有圆盘粘合操作的图对应的曲面添加到顶点 $\mathcal{V}_{g,n}$ 中。

A few facts are simple to understand. At genus zero, we have

有几个结论很容易理解。亏格为零时，我们有:

$$\mathcal{V}_{0,n}^{1\mathrm{PI}} = \mathcal{V}_{0,n} \tag{263}$$

This is because the rest of the relevant space $\mathcal{F}_{0,n}$ in $\widehat{\mathcal{P}}_{0,n}$ arises from tree graphs, thus clearly one-particle reducible. For genus one and one puncture, the full cover $\mathcal{F}_{1,1}$ in $\widehat{\mathcal{P}}_{1,1}$ is produced by adding to $\mathcal{V}_{1,1}$ the surfaces from the one-loop diagram where the three-string vertex $\mathcal{V}_{0,3}$ has two punctures glued together. Since this diagram is 1PI, it is included in the 1PI vertex, which is now seen to be the full cover

这是因为 $\widehat{\mathcal{P}}_{0,n}$ 中其余相关空间 $\mathcal{F}_{0,n}$ 都来自树图，因此显然是单粒子可约的。对于亏格为一且有一个 puncture 的情况，$\widehat{\mathcal{P}}_{1,1}$ 中的完整覆盖 $\mathcal{F}_{1,1}$ 是通过在 $\mathcal{V}_{1,1}$ 的基础上添加单圈图的曲面得到的: 该图中三弦顶点 $\mathcal{V}_{0,3}$ 有两个 puncture 粘合在一起。由于这个图是 1PI 的，它会被包含在 1PI 顶点中，此时 1PI 顶点就是完整覆盖。

$$\mathcal{V}_{1,1}^{1\mathrm{PI}} = \mathcal{F}_{1,1} \tag{264}$$

More generally, $\mathcal{V}_{g,n}^{1PI}$ includes $\mathcal{V}_{g,n}$ plus a subset of surfaces with one disk-gluing operation, two disk-gluing operations, and so on, with the constraint that cutting along any gluing curve does not split the Riemann surface. Surfaces obtained with one disk-gluing operation clearly take the vertex $\mathcal{V}_{g-1,n+2}$ and disk-glue two of the punctures. One can also describe explicitly the structure of configurations with two and more disk-gluing operations. While this construction does not refer to gauge-fixing, it should be noted that in Siegel gauge, amplitudes are constructed exactly via disk-gluing. It is thus possible that alternative constructions of the 1PI action could be obtained in other string field theory gauges.

更一般地说，$\mathcal{V}_{g,n}^{1PI}$ 包含 $\mathcal{V}_{g,n}$，再加上满足“沿任意粘合曲线切割都不会拆分黎曼曲面”约束的、带有一次、两次……圆盘粘合操作的曲面子集。经过一次圆盘粘合操作得到的曲面显然是取顶点 $\mathcal{V}_{g-1,n+2}$ 并将其中两个 puncture 圆盘粘合得到的。我们也可以明确描述带有两次及以上圆盘粘合操作的构型结构。虽然这个构建过程不涉及规范固定，但需要注意，西格尔规范中振幅正是通过圆盘粘合构建的。因此，有可能在其他弦场论规范中得到 1PI 作用量的替代构建方案。

Having constructed the $\mathcal{V}_{g,n}^{1\mathrm{PI}}$, we define the formal sum

构造出 $\mathcal{V}_{g,n}^{1\mathrm{PI}}$ 后，我们定义形式和

$$\mathcal{V}_{1\mathrm{PI}} \equiv \sum_{g,n} \mathcal{V}_{g,n}^{1\mathrm{PI}} \tag{265}$$

This object satisfies the classical version of the geometric master equation, namely,

该对象满足几何主方程的经典形式，即

$$\partial \mathcal{V}_{1\mathrm{PI}} + \frac{1}{2}\{\mathcal{V}_{1\mathrm{PI}}, \mathcal{V}_{1\mathrm{PI}}\} = 0, \tag{266}$$

reflecting that tree diagrams constructed from the 1PI effective action give a space $\mathcal{F}_{g,n} \subset \widehat{\mathcal{P}}_{g,n}$, which is a cover of $\mathcal{M}_{g,n}$. To show that the equation above holds, we first argue that the surfaces in $\partial \mathcal{V}_{1\mathrm{PI}}$ coincide with those in $\{\mathcal{V}_{1\mathrm{PI}}, \mathcal{V}_{1\mathrm{PI}}\}$. Surfaces in $\partial \mathcal{V}_{1\mathrm{PI}}$ do not arise from the disk-gluing operation, which does not create boundaries. Indeed, as the disk-gluing parameter $q \to 0$, we get nodal surfaces, which are a codimension two subspace relative to the vertex dimension, nor are surfaces arising from $|q| = 1$ boundaries of the vertex region, for here two vertices join to form a "bigger" vertex whose moduli space will include surfaces where the gluing curve becomes larger. Surfaces in $\partial \mathcal{V}_{1\mathrm{PI}}$ must arise when some vertex within $\mathcal{V}_{1\mathrm{PI}}$ reaches a boundary. When a vertex reaches a boundary, the surface $\sum$ in the vertex must have a curve that acquires the critical length and can play the role of a gluing curve. If cutting the gluing curve does not split the whole surface of the diagram, this is not a boundary; this curve can be used for disk-gluing, and a diagram with such gluing is included in the full 1PI vertex. A boundary is genuine only if the curve on the surface $\sum$, if cut, would split the whole surface in the diagram in two pieces. If this happens, each piece must be 1PI since, otherwise, the whole diagram would have not been 1PI. Such a boundary is precisely reproduced by $\{\mathcal{V}_{1\mathrm{PI}}, \mathcal{V}_{1\mathrm{PI}}\}$. Now we argue that any surface in $\{\mathcal{V}_{1\mathrm{PI}}, \mathcal{V}_{1\mathrm{PI}}\}$ is 1PI at the boundary $\partial \mathcal{V}_{1\mathrm{PI}}$. It is clear that it is 1PI because each piece and the only new curve generated by the twist-gluing is not a propagator; it is not disk-gluing. The twist-gluing operation in this bracket creates at the joint a "larger" vertex with a critical length gluing curve that separates. That means, as seen above, that these are surfaces in $\partial \mathcal{V}_{1\mathrm{PI}}$. All in all, the two terms in (266) contain the same surfaces, and this completes our argument.

这说明由 1PI 有效作用量构造的树图给出空间 $\mathcal{F}_{g,n} \subset \widehat{\mathcal{P}}_{g,n}$ ，它是 $\mathcal{M}_{g,n}$ 的覆盖。为证明上述等式成立，我们首先论证 $\partial \mathcal{V}_{1\mathrm{PI}}$ 中的曲面与 $\{\mathcal{V}_{1\mathrm{PI}}, \mathcal{V}_{1\mathrm{PI}}\}$ 中的曲面一致。$\partial \mathcal{V}_{1\mathrm{PI}}$ 中的曲面并非来自圆盘粘合操作——该操作不会产生新边界。实际上，当圆盘粘合参数取 $q \to 0$ 时，我们得到结点曲面，它相对于顶点维数是余维为 2 的子空间；$\partial \mathcal{V}_{1\mathrm{PI}}$ 中的曲面也并非来自顶点区域的 $|q| = 1$ 边界，因为这里两个顶点拼接会形成一个“更大”的顶点，其模空间会包含粘合曲线变大后的曲面。$\partial \mathcal{V}_{1\mathrm{PI}}$ 中的曲面必然出现在 $\mathcal{V}_{1\mathrm{PI}}$ 内某个顶点到达边界时。当顶点到达边界时，该顶点内的曲面 $\sum$ 上一定存在一条获得临界长度的曲线，可充当粘合曲线。如果切割这条粘合曲线没有拆分整个图的曲面，这就不是真正边界；该曲线可用于圆盘粘合，带这种粘合的图已经包含在完整 1PI 顶点中。只有切割曲面 $\sum$ 上的曲线会将图中整个曲面拆分为两部分时，才是真正边界。这种情况下，每一部分都必须是 1PI 的，否则整个图本身就不是 1PI 的。这类边界恰好可以由 $\{\mathcal{V}_{1\mathrm{PI}}, \mathcal{V}_{1\mathrm{PI}}\}$ 重现。现在我们论证 $\{\mathcal{V}_{1\mathrm{PI}}, \mathcal{V}_{1\mathrm{PI}}\}$ 中的任意曲面都位于 $\partial \mathcal{V}_{1\mathrm{PI}}$ 边界的 1PI 。显然它是 1PI 的，因为每个部分都是 1PI，且扭粘合产生的唯一新曲线不是传播子，也不是圆盘粘合。这个对易子中的扭粘合操作在拼接处生成一个“更大”的顶点，带有一条可分离的临界长度粘合曲线。如上所述，这说明这些曲面都属于 $\partial \mathcal{V}_{1\mathrm{PI}}$ 。总而言之，(266) 中的两项包含相同曲面，我们的论证完成。

The one-particle irreducible vertices can now be used to define multilinear maps to the Grassmann algebra as usual:

现在可以按照常规方法，用单粒子不可约顶点定义到格拉斯曼代数的多重线性映射:

$$\{A_1, \cdots, A_n\}_{1\mathrm{PI}} = \sum_{g=0}^{\infty} g_s^{2g+n-2} \int_{\mathcal{V}_{g,n}^{1\mathrm{PI}}} \Omega_{6g-6+2n}^{(g,n)}(A_1, \cdots, A_n). \tag{267}$$

Then the 1PI effective action of heterotic or type II string theory is given by

杂化弦论或 II 型弦论的 1PI 有效作用量由下式给出

$$S_{1\mathrm{PI}} = -\frac{1}{2}\langle\widetilde{\Psi}, Q\mathcal{G}\widetilde{\Psi}\rangle + \langle\widetilde{\Psi}, Q\Psi\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\{\Psi^n\}_{1\mathrm{PI}}. \tag{268}$$

The corresponding action for the bosonic string field theory is obtained by setting $\widetilde{\Psi} = \Psi$ as well as $\mathcal{G} = 1$
.

玻色弦场论的对应作用量可通过设置 $\widetilde{\Psi} = \Psi$ 和 $\mathcal{G} = 1$ 得到。

Since we are supposed to only compute tree amplitudes with the 1PI effective action, we can restrict the string fields to have ghost number corresponding to classical string fields. The corresponding 1PI action will have a gauge invariance analogous to (131)

由于我们只需用 1PI 有效作用量计算树振幅，因此可以将弦场限制为对应经典弦场的鬼数。对应的 1PI 作用量具有类似 (131) 的规范不变性

$$\delta\,|\widetilde{\Psi}\rangle = Q|\,\widetilde{\Lambda}\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}[\Lambda\Psi^n]_{1\mathrm{PI}},\ \delta\,|\Psi\rangle = Q|\,\Lambda\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{G}[\Lambda\Psi^n]_{1\mathrm{PI}},$$

(269)

We can also relax the constraint on the ghost number and interpret $S_{1\mathrm{PI}}$ as the 1PI effective action of the full quantum string field. In this case, it will satisfy classical BV master equation:

我们也可以放松对鬼数的约束，将 $S_{1\mathrm{PI}}$ 解释为全量子弦场的 1PI 有效作用量。这种情况下，它满足经典 BV 主方程:

$$\{S_{1\mathrm{PI}}, S_{1\mathrm{PI}}\} = 0. \tag{270}$$

Wilsonian Effective Action

威尔森有效作用量

In quantum field theory, we have a notion of effective action, where we integrate out a subset of the fields and write down an effective action involving the remaining fields. A similar construction exists in string field theory [69]. Interestingly, the effective field theory for a selected subset of the fields is governed by the same algebraic identities as those of the parent string field theory. We shall illustrate this in the context of type II and heterotic string field theory.

在量子场论中，我们有有效作用量的概念: 我们积掉一部分场，写出仅含剩余场的有效作用量。弦场论中也存在类似构造 [69]。有趣的是，选定子集场的有效场论满足与原弦场论相同的代数恒等式。我们将在 II 型和杂化弦场论的框架下对此进行说明。

The chosen set of fields, the ”light fields,” is selected by the use of a projection operator P acting on the $\mathcal{H}_c$ and $\widetilde{\mathcal{H}}_c$ subspaces used earlier to formulate the parent string field theory (of type II or heterotic strings). ”Heavy fields” are selected by the complementary projector $1-P$. For consistency of the construction below, the projector P is required to satisfy a number of conditions:

我们选取的这组场称为“轻场”，通过投影算符 P 作用在之前构建原弦场论 (II 型或杂化弦的) 时用到的 $\mathcal{H}_c$ 和 $\widetilde{\mathcal{H}}_c$ 子空间选出。“重场”则由互补投影算符 $1-P$ 选出。为保证下文构造的自洽性，要求投影算符 P 满足若干条件:

$$\left[P, L_0^{\pm}\right]=0,\ \left[P, b_0^{\pm}\right]=0,\ \left[P, c_0^{\pm}\right]=0,\ [P, Q]=0,\ [P, \mathcal{G}]=0.$$

(271)

The main ingredient in the construction of the parent string field theory was a set of multilinear maps $\{\cdots\}$ to the Grassmann algebra defining vertices for the string field in $\mathcal{H}_c$. Now we require a new set of multilinear maps defining vertices for the light fields-those in $P\mathcal{H}_c$. We write those as follows:

原弦场论构造的核心是一组到格拉斯曼代数的多重线性映射 $\{\cdots\}$，用于定义 $\mathcal{H}_c$ 中弦场的顶点。现在我们需要一组新的多重线性映射，来定义轻场 (即 $P\mathcal{H}_c$ 中的场) 的顶点，具体形式如下:

$$\left\{A_1, \cdots, A_n\right\}_{\text{eff}},\ A_1, \cdots, A_n \in P\mathcal{H}_c. \tag{272}$$

These effective multilinear maps represent the calculation of a modified version of the off-shell amplitude for the states $A_1, \cdots, A_n$. At each genus g (and n punctures), this amplitude requires contributions from all the surfaces in the moduli space $\mathcal{M}_{g,n}$, with a simple modification. The contribution from the vertex $\mathcal{V}_{g,n}$ is left unchanged. The contribution from the Feynman diagrams using lower-order vertices and internal propagators is changed by inserting the projector $1-P$ to heavy fields on each propagator. This in effect allows only the heavy fields to run on the internal lines. This is a natural prescription in an effective theory where the kinetic term will only propagate light fields.

这些有效多重线性映射对应修正后的离壳振幅计算，对象是态 $A_1, \cdots, A_n$ 。对任意亏格 g (带 n 个 puncture)，该振幅仅需对模空间 $\mathcal{M}_{g,n}$ 中的所有曲面做简单修正即可得到贡献: 顶点 $\mathcal{V}_{g,n}$ 的贡献保持不变；使用低阶顶点和内部传播子的费曼图，修正方式是在每个传播子上插入投影到重场的投影算符 $1-P$ ，这实际上仅允许重场在内线传播。这是有效理论中自然的处理方式，因为动能项仅传播轻场。

Then the effective action is an action for string fields

因此，有效作用量就是轻弦场的作用量

$$\Psi \in P\mathcal{H}_c,\ \widetilde{\Psi} \in P\widetilde{\mathcal{H}}_c. \tag{273}$$

and is given by

其形式为

$$S_{\text{eff}} = -\frac{1}{2}\langle\widetilde{\Psi}, Q\mathcal{G}\widetilde{\Psi}\rangle + \langle\widetilde{\Psi}, Q\Psi\rangle + \sum_n \frac{1}{n!}\{\Psi^n\}_{\text{eff}}. \tag{274}$$

One can show that this action satisfies the quantum BV master equation with the symplectic structure (129), restricted to subspaces $P\mathcal{H}_c$ and $P\widetilde{\mathcal{H}}_c$.

可以证明，这个作用量满足限制在子空间 $P\mathcal{H}_c$ 和 $P\widetilde{\mathcal{H}}_c$ 上、带辛结构 (129) 的量子 BV 主方程。

For a given state in $\mathcal{H}_c$, let $L_0', \bar{L}_0'$ denote the contribution to $L_0, \bar{L}_0$ other than the momentum contribution $k^2/4$. Thus, in string theory in flat space-time, $L_0', \bar{L}_0'$ are the total number operators. If P is taken to be the projection operator into the $L_0'^+ = 0$ states, it is a projector into states with $L_0' = 0$ and $\bar{L}_0' = 0$, due to the (off-shell) vanishing of $L_0^- = L_0'^-$. Thus, P is a projector to massless states, and all the massive states are integrated out. The massless states, however, can have high momentum, so the action, as given, is not the familiar Wilsonian effective action. A modification of the vertices, however, can make the action accurately Wilsonian, as we explain now.

对 $\mathcal{H}_c$ 中的任意给定态，设 $L_0', \bar{L}_0'$ 表示除动量贡献 $k^2/4$ 外对 $L_0, \bar{L}_0$ 的贡献，因此在平直时空弦论中，$L_0', \bar{L}_0'$ 就是总粒子数算符。若取 P 为投影到 $L_0'^+ = 0$ 态的投影算符，由于 $L_0^- = L_0'^-$ (离壳) 为零，它就是投影到满足 $L_0' = 0$ 和 $\bar{L}_0' = 0$ 态的投影算符，因此 P 就是到无质量态的投影算符，所有有质量态都被积掉了。但无质量态仍可拥有高动量，因此上述形式的作用量并不是我们熟悉的威尔森有效作用量。我们接下来会说明，只要对顶点做一处修改，就能得到标准的威尔森有效作用量。

Suppose we choose to modify the chains $\mathcal{V}_{g,n}$ in such a way that the local coordinates w at the punctures are scaled by a large positive factor λ , i.e., the new local coordinates $\widetilde{w}$ are such that $|\widetilde{w}| = 1$ corresponds to $|w| = 1/\lambda$. In that case, the contribution from the vertex for off-shell external states acquires suppression factors proportional to λ^{-h_i} , where h_i is the L_0^+ eigenvalue of the i -th external state. For $L_0'^+ = 0$ states, this translates to a factor of $\lambda^{-k^2/2}$, showing that the contribution from large k^2 modes are suppressed in the Feynman diagrams. Effectively, the large k^2 modes have been integrated out. This now matches the definition of a Wilsonian effective action, where we integrate out the massive fields and the high momentum modes of the massless fields. Therefore, the effective action, obtained after integrating out all the $L_0'^+ > 0$ modes and choosing $\mathcal{V}_{g,n}$ where local coordinates at the punctures have large-scale factors, gives the Wilsonian effective action of string field theory.

假设我们选择修改链 $\mathcal{V}_{g,n}$ ，使得孔点处的局部坐标 w 被一个大的正因子 λ 缩放，即新局部坐标 $\widetilde{w}$ 满足 $|\widetilde{w}| = 1$ 对应 $|w| = 1/\lambda$ 。在这种情况下，脱壳外态顶点的贡献会获得与 λ^{-h_i} 成正比的压制因子，其中 h_i 是第 i 个外态的 L_0^+ 本征值。对于 $L_0'^+ = 0$ 态，这转化为因子 $\lambda^{-k^2/2}$ ，表明大 k^2 模式的贡献在费曼图中被压制。实际上，大 k^2 模式已经被积掉了。这刚好匹配威尔逊有效作用量的定义: 我们积掉有质量场和无质量场的高动量模式。因此，积掉所有 $L_0'^+ > 0$ 模式后、选择孔点局部坐标带大尺度因子的 $\mathcal{V}_{g,n}$ 得到的有效作用量，就是弦场论的威尔逊有效作用量。

The above scaling of local coordinates can be interpreted as the inclusion of stubs to the surface. To see this, first note that the local coordinate w , defined for the unit disk $|w| \le 1$, can be viewed as a semi-infinite cylinder of circumference 2π via the map $z = \ln w$ that takes the disk into the strip $0 \leq \mathrm{Im}(z) \leq 2\pi$ and $\mathrm{Re}(z) \leq 0$ with the horizontal boundaries identified. Then the retraction of the local coordinate disk to $|w| = 1/\lambda$ corresponds to adding to the surface the annulus $1/\lambda < |w| < 1$. In the cylinder picture, we are adding a stub of length $\ln\lambda$: a short cylinder of circumference 2π and height $\ln\lambda$. Stubs were introduced in closed string field theory to define local coordinates that made minimal area metrics consistent under the operation of gluing [6], as will be reviewed in section "Minimal Area String Vertices: Witten Vertex and Closed String Polyhedra".

上述局部坐标缩放可以理解为给世界面添加短线段 (stub)。为说明这一点，首先注意，定义在单位圆盘 $|w| \leq 1$ 上的局部坐标 w ，可以通过映射 $z = \ln w$ 视为周长为 2π 的半无限圆柱: 该映射将圆盘映射到带粘合水平边界的带状区域 $0 \leq \mathrm{Im}(z) \leq 2\pi$ 和 $\mathrm{Re}(z) \leq 0$ 。那么将局部坐标圆盘回缩到 $|w| = 1/\lambda$ ，就对应给世界面添加环形区域 $1/\lambda < |w| < 1$ 。在圆柱图像中，我们就是添加了长度为 $\ln\lambda$ 的短线段: 一个周长为 2π 、高度为 $\ln\lambda$ 的短圆柱。引入短线段是为了在闭弦场论中定义局部坐标，使极小面积度规在粘合操作下保持一致性 [6]，我们会在“极小面积弦顶点: 威滕顶点与闭弦多面体”一节回顾这点。

The notion of string theory effective actions has been usefully systematized in the context of homotopy transfer [70-74] of A_∞ and L_∞ structures for open and closed string theories, respectively. Under a number of precise conditions, A_∞ and L_∞ algebras on a "parent" state space give rise to derived A_∞ and L_∞ structures on projected spaces (see section "Homotopy Transfer"). Explicit formulae for the products for the derived algebras are available. Applications and extensions of these ideas can be found in [75, 76] .

弦论有效作用量的概念已经在同伦转移 [70-74] 的框架下得到了有效的系统化，分别对应开弦和闭弦论的 A_∞ 结构与 L_∞ 结构。在若干精确条件下，“母”态空间上的 A_∞ 代数和 L_∞ 代数会在投影空间上诱导出导出 A_∞ 结构和 L_∞ 结构 (参见“同伦转移”一节)，目前已经得到了导出代数乘积的显式公式，这些想法的应用与推广可以参见 [75, 76] 。

Equivalence of Different String Field Theories

不同弦场论的等价性

As we have seen, the construction of the interaction vertices in string field theory enjoys a lot of freedom, encoded in the choice of local coordinates at the punctures, which, in turn, determines how the whole moduli space of Riemann surfaces is covered by different Feynman diagrams of string field theory. Given this, one could wonder how the different string field theories are related. The simple answer to this question is that they are all related to each other by field redefinition. In this section, we shall briefly describe how one can prove this.

如我们所见，弦场论中相互作用顶点的构造拥有很大自由度，这体现在对孔点处局部坐标的选择中，而局部坐标的选择反过来又决定了黎曼曲面整个模空间如何被弦场论不同费曼图覆盖。有鉴于此，人们自然会好奇不同弦场论之间有何关联。这个问题的简单答案是: 它们全部通过场重新定义相互关联。本节我们将简要介绍如何证明这一结论。

In general, a field redefinition will affect the form of the action and also the integration measure, and one needs to keep track of both to show the equivalence between two different formulations of string field theory.

To avoid this, we can work with the 1PI irreducible effective action where all the quantum corrections are already included in the action. In this case, the proof of equivalence of two different formulations of string field theory just involves the existence of a field redefinition that preserves the antibracket and leaves the action invariant [77].

一般而言，场重新定义会作用量的形式和积分测度，因此要证明两种不同弦场论表述等价，必须同时兼顾两者。为了规避这一点，我们可以使用已经包含所有量子修正的 1PI 不可约有效作用量进行研究。在这种情况下，两种不同弦场论表述等价性的证明只需证明: 存在一个保持反括号且让作用量不变的场重新定义 [77]。

For simplicity, we shall focus on closed bosonic string field theory and follow the approach of Hata and one of us [78], which can be generalized to open string theory as well as superstring field theory. Let us consider a 1PI effective string field theory whose vertices are described by $\mathcal{V}_{g,n}^{1\text{PI}} \subset \widehat{\mathcal{P}}_{g,n}$, for various values of the genus g and the number of punctures n . Then the formal sum

为简化讨论，我们将聚焦闭玻色弦场论，遵循畑和本文作者之一提出的方法 [78]，该方法可以推广到开弦理论和超弦场论。我们考虑一个 1PI 有效弦场论，其顶点由 $\mathcal{V}_{g,n}^{1\text{PI}} \subset \widehat{\mathcal{P}}_{g,n}$ 描述，对应亏格 g 和孔数 n 的各种取值。那么形式和

$$\mathcal{V}^{1\text{PI}} = \sum_{g,n} g_s^{2g-n+2} \mathcal{V}_{g,n}^{1\text{PI}}, \tag{275}$$

satisfies

满足

$$\partial \mathcal{V}^{1\text{PI}} + \frac{1}{2}\left\{\mathcal{V}^{1\text{PI}}, \mathcal{V}^{1\text{PI}}\right\} = 0, \tag{276}$$

Let $\mathcal{V}_{g,n}'^{1\text{PI}} \subset \widehat{\mathcal{P}}_{g,n}$ be another set of vertices for 1PI effective field theory that are infinitesimally close to the original set of vertices $\mathcal{V}_{g,n}^{1\text{PI}}$. Then the 1PI actions $S_{1\text{PI}}$ and $S'_{1\text{PI}}$ constructed from these two sets of vertices differ by

设 $\mathcal{V}_{g,n}'^{1\text{PI}} \subset \widehat{\mathcal{P}}_{g,n}$ 是 1PI 有效场论的另一组顶点，与原顶点组 $\mathcal{V}_{g,n}^{1\text{PI}}$ 无穷小邻近。那么由这两组顶点分别构造的 1PI 作用量 $S_{1\text{PI}}$ 和 $S'_{1\text{PI}}$ 的差为

$$S'_{1\text{PI}} - S_{1\text{PI}} = \sum_{g=0}^{\infty}\sum_{n=1}^{\infty} g_s^{2g-n+2}\left(\int_{\mathcal{V}_{g,n}'^{1PI}} - \int_{\mathcal{V}_{g,n}^{1\text{PI}}}\right)\Omega_{6g-6+2n}^{(g,n)}\left(\Psi^{\otimes n}\right). \tag{277}$$

In order to show that $S_{1\text{PI}}$ and $S'_{1\text{PI}}$ are related by a field redefinition, we need to show that there is a field redefinition that relates the two actions. In other words, under such field redefinition, the change in $S_{1\text{PI}}$ should be given by $S'_{1\text{PI}} - S_{1\text{PI}}$.

要证明 $S_{1\text{PI}}$ 和 $S'_{1\text{PI}}$ 可以通过场重新定义联系起来，我们需要证明存在一个场重新定义可以关联这两个作用量。换句话说，在该场重新定义下，$S_{1\text{PI}}$ 的变化应当由 $S'_{1\text{PI}} - S_{1\text{PI}}$ 给出。

Such field redefinitions can be constructed explicitly. In order to keep the discussion short, we shall give the result without giving the derivation. Let $\widehat{U}_{g,n}$ be an infinitesimal vector in the neighborhood of $\mathcal{V}^{\mathrm{1PI}}_{g,n}$ in $\widehat{\mathcal{P}}_{g,n}$ that takes a point on $\mathcal{V}^{\mathrm{1PI}}_{g,n}$ to a neighboring point on $\mathcal{V}'^{1PI}_{g,n}$. Clearly $\widehat{U}_{g,n}$ is ambiguous, since we can add to $\widehat{U}_{g,n}$ a tangent vector of $\mathcal{V}^{\mathrm{1PI}}_{g,n}$, but this ambiguity will not affect our result. Now one can show that under a field redefinition $\Psi \to \Psi + \delta\Psi$, with $\delta\Psi$ defined via the equation

我们可以显式构造出这样的场重新定义。为了简化讨论，我们在此只给出结果而省略推导。设 $\widehat{U}_{g,n}$ 是 $\widehat{\mathcal{P}}_{g,n}$ 空间中 $\mathcal{V}^{\mathrm{1PI}}_{g,n}$ 邻域内的无穷小向量，它将 $\mathcal{V}^{\mathrm{1PI}}_{g,n}$ 上的一点映射到 $\mathcal{V}'^{1PI}_{g,n}$ 上相邻的一点。显然 $\widehat{U}_{g,n}$ 存在歧义，因为我们可以给 $\widehat{U}_{g,n}$ 加上一个 $\mathcal{V}^{\mathrm{1PI}}_{g,n}$ 的切向量，但这种歧义不影响我们的结果。现在可以证明，在场重新定义 $\Psi \to \Psi + \delta\Psi$ 下，其中 $\delta\Psi$ 由下式定义

$$\langle \Phi | c_0^- | \delta\Psi \rangle = -\sum_{g=0}^{\infty}\sum_{n=1}^{\infty} g_s^{2g-n+2}\frac{1}{(n-1)!}\int_{\mathcal{V}^{\mathrm{1PI}}_{g,n}} \Omega^{(g,n)}_{6g-5+2n}\left(\Phi, \Psi^{\otimes(n-1)}\right)\left[\widehat{U}_{g,n}\right],$$

(278)

the change in S_{1PI} precisely gives (277). Here Φ is any state in $\mathcal{H}_c$ defined in (99) and $\left[\widehat{U}_{g,n}\right]$ denotes contraction of the $(6g-5+2n)$ form with the tangent vector $\widehat{U}_{g,n}$, resulting in a $(6g-6+2n)$ form to be integrated over the 1PI vertex.

S_{1PI} 的变化恰好给出式 (277)。此处 Φ 是式 (99) 中定义的 $\mathcal{H}_c$ 内任意状态，$\left[\widehat{U}_{g,n}\right]$ 表示 $(6g-5+2n)$ 形式与切向量 $\widehat{U}_{g,n}$ 的缩并，缩并后得到一个将要在 1PI 顶点上积分的 $(6g-6+2n)$ 形式。

The construction of [78] was at the infinitesimal level, and the uniqueness was shown assuming that the vertices are submanifolds and partial sections of $\widehat{\mathcal{P}}_{g,n}$. These limitations were overcome in [79], as explained in detail in [34]. We will touch on this in section ”Geometric BV Master Equation and String Field Theory Master Equation”, when discussing the uniqueness of solutions of the geometric master equation.

文献 [78] 中的构造是无穷小层次的，并且在假设顶点是 $\widehat{\mathcal{P}}_{g,n}$ 的子流形和局部截面的前提下证明了唯一性。正如文献 [34] 中详细解释的，这些限制已经在文献 [79] 中被克服。我们会在“几何 BV 主方程与弦场论主方程”一节讨论几何主方程解的唯一性时提及这部分内容。

Background Independence

背景无关性

We have seen that the construction of string field theory requires us to start with a given background, encoded in the world-sheet (super-)conformal field theory underlying the string theory, and then construct the interaction vertices of string field theory using the correlation functions in that particular world-sheet (super-)conformal field theory. However, on physical grounds, we expect that string theories constructed around different backgrounds should be related to each other by an appropriate redefinition of the fields. [10] This has been established in string field theory when the backgrounds are related by (infinitesimal) marginal deformation. In this section, we shall briefly describe how this is done.

我们已经看到，弦场论的构建要求我们从给定背景出发，该背景编码在弦理论 underlying 的世界面 (超) 共形场论中，随后利用该特定世界面 (超) 共形场论中的关联函数构建弦场论的相互作用顶点。然而从物理角度出发，我们预期围绕不同背景构建的弦理论应当可以通过恰当的场重新定义相互关联。[10] 对于由 (无穷小) 边缘形变联系起来的背景，这一点已经在弦场论中得到证实。本节我们将简要介绍这一结论的推导过程。

The proof of background independence of the classical theory involves showing that the classical action formulated around two different backgrounds are related by field redefinition. This was proved in [80] following earlier work of [81-83]. The proof of background independence of the quantum theory is more delicate; this time, the combination of the action and the integration measure must be shown to be background independent, and this was achieved in [84]. As in section ”Equivalence of Different String Field Theories”, we can avoid this complication by using the 1PI effective action, where the steps involved in the proof are simpler to explain. Also, let us first focus on closed bosonic string field theory for definiteness. Under infinitesimal marginal deformation, the BRST operator as well as all correlation functions of the CFT get deformed. This changes the kinetic term and the interaction terms of string field theory. Let us denote the 1PI effective action of string field theory around the deformed background by $S'_{1\text{PI}}$. Then the problem of showing background dependence boils down to showing that there is a deformation $\delta\Psi$ of the string field under which the change in $S_{1\text{PI}}$ is precisely equal to $S'_{1\text{PI}} - S_{1\text{PI}}$. An explicit form of $\delta\Psi$ satisfying this property can be found in [85].

经典理论背景无关性的证明需要表明, 在两个不同背景下构造的经典作用量可通过场重新定义相互联系，这一结论在 [80] 中基于 [81-83] 的前期工作得到证明。量子理论背景无关性的证明则更为精细: 这一次需要证明作用量与积分测度的组合是背景无关的，该工作在 [84] 中完成。和“不同弦场论的等价性”一节一样，我们可以利用 1PI 有效作用量规避这类复杂性，1PI 有效作用量下证明的步骤更便于阐述。此外，为明确起见我们首先聚焦于闭玻色弦场论。在无穷小边缘形变下，BRST 算符以及共形场论的所有关联函数都会发生形变，这会改变弦场论的动能项和相互作用项。我们将形变背景下弦场论的 1PI 有效作用量记为 $S'_{1\text{PI}}$ 。此时证明背景依赖性的问题就归约为: 证明存在弦场的形变 $\delta\Psi$ ，在该形变下 $S_{1\text{PI}}$ 的变化恰好等于 $S'_{1\text{PI}} - S_{1\text{PI}}$ 。满足该性质的 $\delta\Psi$ 的显式形式可在文献 [85] 中找到。

The generalization of the result to open or open-closed bosonic string field theory is straightforward. For superstring field theories, there is an additional subtlety. We have seen that the formulation of string field theory for the Ramond sector fields requires doubling the number of degrees of freedom, but one linear combination of the two fields remains free and decouples from the S-matrix. It turns out that if we deform the original world-sheet SCFT by a marginal operator, then both the free field part of the equations of motion and the interacting field equations change, since both use the deformed BRST operator at the linearized level, and the interacting part of the field also uses the deformed world-sheet correlation functions. On the other hand, if we start with the string field theory formulated in the original background, then by making a suitable redefinition of the fields, we can make the interacting part of the field equations agree with the interacting part of the field equations formulated in the deformed CFT, but the free field equations of motion remain unchanged and continue to use the BRST operator of the original CFT [85]. Therefore, actions formulated around the original background and the deformed background are not related by a field redefinition. However, the agreement between the interacting part of the field equations derived from the 1PI action after field redefinition implies that the S-matrix elements computed using string field theory in the deformed background and string field theory in the original background after field redefinition including shift are identical. This can be taken as the statement of background independence of the theory.

将该结果推广到开弦或开-闭玻色弦场论是十分直接的。对于超弦场论，则存在额外的微妙之处。我们已经知道，拉蒙德 sector 场的弦场论表述要求加倍自由度的数量，但两个场的一个线性组合始终是自由的，并且退耦合于 S 矩阵。结果表明，如果我们用边缘算符形变原始的世界面超共形场论，运动方程的自由场部分和相互作用场方程都会发生改变，因为二者在线性化层面都使用形变后的 BRST 算符，同时场的相互作用部分也会使用形变后的世界面关联函数。另一方面，如果我们从原始背景下表述的弦场论出发，通过对场做合适的重新定义，我们可以让场方程的相互作用部分与形变后共形场论中表述的场方程相互作用部分一致，但自由场运动方程保持不变，继续使用原始共形场论的 BRST 算符 [85]。因此，围绕原始背景和形变背景构造的作用量无法通过场重新定义联系起来。然而，场重新定义后从 1PI 作用量导出的场方程相互作用部分的一致性意味着，在形变背景的弦场论中计算得到的 S 矩阵元，与原始背景弦场论经过包含平移的场重新定义后计算得到的 S 矩阵元完全一致，这可以作为该理论背景无关性的表述。

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[10] The usual lore is that a field redefinition does not change the S-matrix. However, the field redefinition that relates string field theories around different backgrounds involves a shift that changes the vacuum of the theory. As a result, the S-matrix changes.

[10] 通常的说法是，场重新定义不会改变 S 矩阵。但联系不同背景下弦场论的场重新定义包含会改变理论真空的平移，因此 S 矩阵会发生改变。

The analysis can also be generalized to the case where we have a pair of CFTs connected by a nearly marginal deformation [86,87]. In this case, the central charges of the two CFTs differ, but this can be compensated by having a linear dilaton background field.

该分析也可以推广到由近边缘形变连接一对共形场论的情况 [86,87]。在这种情况下，两个共形场论的中心荷不同，但这可以通过线性 dilaton 背景场得到补偿。

Dilaton Theorem

伸缩子定理

The dilaton theorem is the statement that a shift of the zero-momentum dilaton in the string field theory action has the effect of changing the string coupling of the theory. Our discussion here will focus on the bosonic closed string field theory, for which this theorem was proven in [88,89]. The proof applies for any closed string background because the zero momentum dilaton state can be represented by a universal state built up from ghost operators only and thus is always present; this is the so-called ghost dilaton $D(z,\bar{z})$ given by

伸缩子定理指出，弦场论作用量中零动量伸缩子移动的效应是改变该理论的弦耦合。我们此处的讨论将聚焦于玻色闭弦场论，该定理已在文献 [88,89] 中得到证明。证明适用于任意闭弦背景，因为零动量伸缩子态可表示为仅由鬼算符构造的普适态，因此始终存在；这就是所谓的鬼伸缩子 $D(z,\bar{z})$，由下式给出

$$D\left(z,\bar{z}\right)=\frac{1}{2}\left(c\partial^2c-\bar{c}\bar{\partial}^2\bar{c}\right). \tag{279}$$

The associated state $|D\rangle$ obtained by acting with this field on the vacuum $|\,0\rangle$ is

关联态 $|D\rangle$ obtained by acting with this field on the vacuum $|\,0\rangle$ 为

$$|D\rangle=(c_1c_{-1}-\bar{c}_1\bar{c}_{-1})|\,0\rangle. \tag{280}$$

Note that this state, carrying general momentum, appeared in section ”Kinetic Term for Massless Closed String Fields” as the coefficient of the field d (see Eqs. (208) and (197)). This zero-momentum state is killed by the BRST operator, and it is not BRST trivial; thus, it is a physical state. It also obeys the subsidiary conditions needed to belong to the closed string field theory state space:

请注意，这个携带广义动量的态，已经在“无质量闭弦场的动能项”一节中作为场 d 的系数出现 (见式 (208) 和 (197))。这个零动量态被 BRST 算符零化，且不是 BRST 平凡的；因此它是一个物理态。它也满足属于闭弦场论态空间所需的辅助条件:

$$b_0^-\,|D\rangle=0,\;L_0^-|\,D\rangle=0. \tag{281}$$

The most direct way to see that the state is BRST invariant is by noticing that

证明该态是 BRST 不变的最直接方法是注意到

$$|D\rangle=Qc_0^-|\,0\rangle=-Q\,|\chi\rangle,\,|\,\chi\rangle=-c_0^-|0\rangle. \tag{282}$$

Clearly, $Q\,|D\rangle=0$, but despite appearances, $|\,D\rangle$ is not trivial because the ”gauge parameter” $|\chi\rangle$ does not belong to the clo
$-|0\rangle$ and does not vanish. Another unusual property of the physical state $|D\rangle$ is that it is not primary! In fact, one can verify that

显 然，$Q\,|D\rangle=0$, but despite appearances, $|\,D\rangle$ 不 是 平 凡 的， 因 为 “规 范 参 数” $|\chi\rangle$ does not belong to the closed string field theory state space. Indeed, | $b_0^-\,|\chi\rangle=-b_0^-c_0^-|\,0\rangle\;=\;-|0\rangle$ 并不为零。物理态 $|D\rangle$ 的另一个特殊性质是它不是原生场！实际上，我们可以验证

$$L_1\,|D\rangle=c_0c_1|\,0\rangle,\;\bar{L}_1\,|D\rangle=-\bar{c}_0\bar{c}_1|\,0\rangle. \tag{283}$$

All higher $L_n,\bar{L}_n$ with $n\geq 2$ kill the state $|D\rangle$. Since the state $|\,D\rangle$ while physical is not a dimension zero primary, its insertion is sensitive to the local coordinates used. The state $|\chi\rangle$ is also not primary: one can verify that $L_1|\,\chi\rangle=c_1|0\rangle$ and $\bar{L}_1\,|\chi\rangle=-\bar{c}_1|\,0\rangle$. The subtleties in dealing with correlators of the non-primary D and χ states were discussed in the context of conformal field theory in [90].

所有满足条件 $n\geq 2$ 的更高阶 $L_n,\bar{L}_n$ 都会零化态 $|D\rangle$. Since the state $|\,D\rangle$ ， 而该物理态不是零维原生场，它的插入对所用的局部坐标敏感。态 $|\chi\rangle$ is also not primary: one can verify that $L_1|\,\chi\rangle=c_1|0\rangle$ 和 $\bar{L}_1\,|\chi\rangle=-\bar{c}_1|\,0\rangle$ 。在共形场论的框架下，文献 [90] 已经讨论过处理非原生 D 与 χ 态关联函数的相关细节问题。

The soft dilaton theorem states that the integral of a dilaton insertion in a correlator of on-shell vertex operators and the $\mathcal{B}\left[\partial/\partial v^i\right]$ insertions on a fixed Riemann surface ends up multiplying the correlator by a constant proportional to the Euler number of the surface, plus total derivative terms with respect of the v^i ’ s that integrate to zero when we integrate over the moduli v^i of the original Riemann surface. Note that we are using v^i to denote the moduli of the original punctured Riemann surface to distinguish it from the moduli u^1, u^2 that we shall introduce shortly to label the location of the dilaton vertex operator on the Riemann surface. First, let us review the proof of this result. The strategy is as follows:

软伸缩子定理指出，在固定黎曼曲面上，壳顶点算符和 $\mathcal{B}\left[\partial/\partial v^i\right]$ 插入的关联函数中插入一个伸缩子后对其积分，最终结果等于原关联函数乘上一个与曲面欧拉数成正比的常数，再加上关于 v^i 的全导数项，当我们对原黎曼曲面的模 v^i 积分时，全导数项积分得零。请注意我们使用 v^i 表示原带孔黎曼曲面的模，以区别于我们接下来引入的标记伸缩子顶点算符在黎曼曲面上位置的模 u^1, u^2 。首先，我们来回顾这个结论的证明。思路如下:

1. Let $\Omega_2(D)$ denote the two-form that describes the effect of inserting the dilaton into an on-shell amplitude in the same sense as (47):

1. 令 $\Omega_2(D)$ 为描述将伸缩子插入壳振幅的二阶形式，其意义和式 (47) 一致:

$$\Omega_2(D) = \left(-\frac{1}{2\pi i}\right) du^1 \wedge du^2 \mathcal{B}\left[\frac{\partial}{\partial u^1}\right] \mathcal{B}\left[\frac{\partial}{\partial u^2}\right] D(w=0). \tag{284}$$

If $|\chi\rangle$ had been a regular state in $\mathcal{H}_c$, then using (52), we would get

若 $|\chi\rangle$ 是 $\mathcal{H}_c$ 中的正则态，那么利用式 (52)，我们会得到

$$\Omega_2(D) = -d\Omega_1(\chi) \tag{285}$$

where Ω_1 is the one-form associated with χ insertion

其中 Ω_1 是和 χ 插入关联的一阶形式

$$\Omega_1(\chi) = \left(-\frac{1}{2\pi i}\right)\left(du^1 \mathcal{B}\left[\frac{\partial}{\partial u^1}\right] + du^2 \mathcal{B}\left[\frac{\partial}{\partial u^2}\right]\right) \chi(w=0). \tag{286}$$

We note that (285) does not hold in general because χ is not an element of $\mathcal{H}_c$ and hence $\Omega_1(\chi)$ is not a well-defined one-form. [11] We shall show that if we consider a region of the moduli space where the dilaton is close to one of the external states, then (285) holds in that region up to addition of total derivatives with respect to the moduli v^i of the original Riemann surface, with d interpreted as the exterior derivative in the space spanned by u^1, u^2 . Since we are ignoring the total derivative terms in v^i , this allows us to express the effect of the dilaton insertion as the insertion of

我们注意到 (285) 一般不成立，因为 χ 不是 $\mathcal{H}_c$ 中的元素，因此 $\Omega_1(\chi)$ 不是一个良定义的一元形式。[11] 我们将证明: 若我们考虑模空间中膨胀子靠近一个外态的区域，则在该区域内，(285) 在原黎曼曲面模 v^i 的全导数相加项之外成立，此处 d 被解释为 u^1, u^2 张成空间中的外导数。由于我们忽略 v^i 中的全导数项，因此我们可以将插入膨胀子的效应表示为插入

$$\int_M \Omega_2(D) + \int_{\partial M} \Omega_1(\chi) \tag{287}$$

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[11] Over domains where the phase of local coordinates is globally well defined, however, the form is well defined.

[11] 但在局部坐标相位整体良定义的区域上，该形式是良定义的。

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where M is the region of the Riemann surface that excludes small disks around other punctures.

其中 M 是黎曼曲面中除去其他小孔周围小圆盘的区域。

2. Next we note that the Euler number $\chi(M)$ of a two-dimensional surface M with boundary ∂M is given by

2. 接下来我们注意到，带边界 ∂M 的二维曲面 M 的欧拉数 $\chi(M)$ 由下式给出:

$$\chi(M) = \frac{1}{2\pi}\int_M K^{(2)} + \frac{1}{2\pi}\int_{\partial M} k^{(1)}. \tag{288}$$

Here $K^{(2)}$ is the two-form Gaussian curvature, computed from a conformal metric $ds = \rho|dz|$, and written as $K^{(2)} = K\rho^2 dx \wedge dy$, with $z = x + iy$ and K the Gaussian curvature (the scalar curvature is $R = 2K$). Moreover, $k^{(1)}$ is the geodesic curvature one-form. More explicitly,

此处 $K^{(2)}$ 是由共形度量 $ds = \rho|dz|$ 计算得到的二元高斯曲率，写作 $K^{(2)} = K\rho^2 dx \wedge dy$ ，其中 $z = x + iy$ 和 K 为高斯曲率 (标量曲率为 $R = 2K$)。此外，$k^{(1)}$ 是测地曲率一元形式。更明确地:

$$K^{(2)} = -2i\partial\bar{\partial}\rho dz \wedge d\bar{z},$$

$$k^{(1)} = d\theta_\gamma - i\left[dz\partial\log\rho - d\bar{z}\bar{\partial}\log\rho\right], \tag{289}$$

where the term $d\theta_\gamma$ computes the rotation angle of the tangent vector to the curve $\gamma \in \partial M$. The Euler number of a surface $\sum_{g,n}$ of genus g with n boundaries is $\chi_{g,n} = 2 - 2g - n$.

其中项 $d\theta_\gamma$ 计算曲线 $\gamma \in \partial M$ 切向量的旋转角度。亏格为 g 、带有 n 个边界的曲面 $\sum_{g,n}$ 的欧拉数为 $\chi_{g,n} = 2 - 2g - n$.

3. Finally, we shall show that

3. 最后，我们将证明

$$\Omega_2(D) = -\frac{1}{2\pi}K^{(2)} \text{ and } \Omega_1(\chi) = -\frac{1}{2\pi}k^{(1)}. \tag{290}$$

This, together with (287) and (288), would then establish the soft dilaton theorem for on-shell amplitudes.

结合 (287) 与 (288)，这就可以证明壳上振幅的软膨胀子定理。

Let us begin with the first step, namely, that when the dilaton is close to one of the other vertex operators, we can replace $\Omega_2(D)$ by $-d\Omega_1(\chi)$. In the string field theory, such contributions come from Feynman diagrams with a three-point function involving D , an on-shell vertex operators $c\bar{c}V$, and an off-shell internal state in $\mathcal{H}_c$ connected by a propagator to the rest of the Feynman diagram. We can now write $D = -Q\chi$ and deform the BRST contour away from the location of χ and through $c\bar{c}V$ to act on the off-shell vertex operator and hence on the propagator. The commutator of Q with the b_0^+ term in the propagator $\mathcal{P}_b$ in (225) brings down a factor of L_0^+ and gives

让我们从第一步开始，即当胀子接近其他顶点算符之一时，我们可以用 $-d\Omega_1(\chi)$ 替换 $\Omega_2(D)$ 。在弦场论中，这类贡献来自包含 D 、on-shell 顶点算符 $c\bar{c}V$ ，以及 $\mathcal{H}_c$ 中通过传播子与费曼图其余部分连接的离壳内部态的三点函数费曼图。我们现在可以写出 $D = -Q\chi$ ，然后将 BRST contour 从 χ 的位置变形，穿过 $c\bar{c}V$ 作用于离壳顶点算符，进而作用于传播子。Q 与传播子 $\mathcal{P}_b$ (见式 (225)) 中 b_0^+ 项的对易子会导出一个因子 L_0^+ ，得到

$$[Q, \mathcal{P}_b] = \frac{1}{2\pi} b_0^- \int_0^{2\pi} d\theta \int_0^{\infty} ds \frac{\partial}{\partial s} e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)}. \tag{291}$$

This is a total derivative term that we now argue can be regarded as the insertion of $-d\chi$. The boundary term from $s = \infty$ vanishes since all states that can contribute have $L_0 > 0$,[12] whereas the boundary term at $s = 0$ is the integral of $\Omega_1(\chi)$ along the part of ∂M associated with the θ integral of the collapsed propagator. The action of Q on the rest of the vertex operators vanishes since all other external states are BRST invariant and the anti-commutator of Q with the $\mathcal{B}\left[\partial/\partial v^i\right]$ insertions will generate total derivative with respect to v^i , which will eventually integrate to zero after integration over the v^i ’ s. This shows that $\Omega_2(D)$ can be replaced by $-d\Omega_1(\chi)$ in this region of the moduli space. The same result holds when the dilaton is close to any other external state vertex operators.

这是一个全导数项，我们现在可以证明它可视为 $-d\chi$ 的插入。来自 $s = \infty$ 的边界项为零，因为所有可贡献的态都满足 $L_0 > 0$,[12] ，而 $s = 0$ 处的边界项是 $\Omega_1(\chi)$ 沿 ∂M 与坍缩传播子 θ 积分相关部分的积分。Q 作用于其余顶点算子的结果为零，因为所有其他外态都是 BRST 不变的，且 Q 与 $\mathcal{B}\left[\partial/\partial v^i\right]$ 插入的反对易子会生成相对于 v^i 的全导数，对 v^i 积分后最终结果为零。这表明在模空间的该区域中，$\Omega_2(D)$ 可以替换为 $-d\Omega_1(\chi)$ 。当 dilation 靠近任何其他外态顶点算子时，该结果同样成立。

We now turn to the proof of (290), following [88]. This is somewhat non-trivial since the dilaton vertex operator is not of the form $c\bar{c}V$ for a dimension $(1,1)$ matter primary V , and hence the procedure described in the paragraph following (44) does not work. In particular, the form that needs to be integrated over the moduli space now depends on the choice of local coordinates at the puncture. We will consider some domain on the Riemann surface with a local uniformizer z used to label the points in this domain. We write the relation between z and the local coordinate w around the puncture where the dilaton is inserted as

我们现在遵循文献 [88] 证明式 (290)。这一过程并非平凡，因为 dilaton 顶点算子不满足 $c\bar{c}V$ 的形式 (其中 $c\bar{c}V$ 是维度为 $(1,1)$ 的物质主元 V)，因此 (44) 段后描述的方法不适用。具体来说，需要在模空间积分的形式现在依赖于孔处局部坐标的选择。我们考虑黎曼曲面上的一个区域，该区域使用局部正则坐标 z 标记区域内的点，将 dilaton 插入孔周围的局部坐标 w 与 z 的关系写为:

$$z = F(w;u) = F\left(w;u^1,u^2\right), \tag{292}$$

where we have parameters $u = \left(u^1,u^2\right)$ with u^1 and u^2 real variables encoding the position of the puncture via the complex variable $y(u)$ obtained by setting $w = 0$ in F :

其中我们有参数 $u = \left(u^1,u^2\right)$ ， u^1 和 u^2 为实变量，它们通过令 $w = 0$ 等于 F 得到的复变量 $y(u)$ 编码了孔的位置:

$$y(u) = F(0;u). \tag{293}$$

Given this definition, we can write z as a series expansion in w , thus describing a general F as follows:

根据该定义，我们可以将 z 写为 w 的级数展开，从而得到任意 F 的形式如下:

$$z = F\left(w;u^1,u^2\right) = y(u) + a(u)\,w + \frac{1}{2}b(u)\,w^2 + \frac{1}{3!}c(u)\,w^2 + \mathcal{O}\left(w^4\right), \tag{294}$$

with a,b,c arbitrary complex-valued functions of the parameters u^1 and u^2 .

其中 a,b,c 是参数 u^1 和 u^2 的任意复值函数。

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[12] There are no tachyonic states since in the matter sector, the state that flows along the propagator is a dimension $(1,1)$ primary V or its descendent. In the ghost sector, the state has ghost number $(0,2)$ or $(2,0)$, and hence the state $c\bar{c}V$ also cannot propagate. This also shows that instead of writing $D = -Q\chi$ and deforming the BRST contour, we could use the original expression involving D and place an upper cutoff Λ in the integral over s associated with the propagator. Then the large Λ limit of the integral is finite and reproduces the full integral. This would mean that the sum of all the Feynman diagrams can be approximated by the integral of $\Omega_2(D)$ over the original Riemann surface except for small regions around the original punctures. Physically, this happens because in string field theory, the punctures are represented as semi-infinite cylinders, and cutting out small regions around the punctures corresponds to cutting out the regions near the far end of the cylinders. Since the cylinders are flat, the integral of $\Omega_2(D)$ from these regions vanishes, and there is no need to include the $\Omega_1(\chi)$ boundary integrals to recover the Euler number.

[12] 不存在快子态，因为在物质场部分，沿传播子流动的态是维度为 $(1,1)$ 的主元 V 或其后代；在鬼场部分，该态的鬼数为 $(0,2)$ 或 $(2,0)$，因此 $c\bar{c}V$ 态也无法传播。这也说明，我们无需写出 $D=-Q\chi$ 并变形 BRST 轮廓，而是可以使用包含 D 的原始表达式，对传播子对应的 s 积分加上上限截断 Λ。该积分在 Λ 趋于无穷大时极限有限，可重现完整积分。这意味着所有费曼图的和可以用原始黎曼曲面上除原孔周围小区域外的 $\Omega_2(D)$ 积分来近似。从物理上看，这是因为弦场论中孔表示为半无限圆柱，挖去孔周围的小区域对应于挖去圆柱远端附近的区域。由于圆柱是平坦的，这些区域对 $\Omega_2(D)$ 的积分为零，因此不需要加入 $\Omega_1(\chi)$ 边界积分来恢复欧拉数。

Let us now determine the antighost insertions

现在我们来确定反鬼插入

$$\mathcal{B}\left[\frac{\partial}{\partial u^i}\right]=\oint b(z)\,dz\frac{\partial F}{\partial u^i}+\oint\bar{b}(\bar{z})\,d\bar{z}\frac{\partial\bar{F}}{\partial u^i} \tag{295}$$

$$=-\oint b(w)\,dw\left(\frac{\partial F}{\partial w}\right)^{-1}\frac{\partial F}{\partial u^i}-\oint\bar{b}(\bar{w})\,d\bar{w}\left(\frac{\partial\bar{F}}{\partial\bar{w}}\right)^{-1}\frac{\partial\bar{F}}{\partial u^i},$$

where we passed to the w frame so that the insertions act directly on the operator, itself inserted in w frame. Next we use (294) to evaluate in a power series in w the expressions inside the above integrals. A short calculation gives

其中我们转换到了 w 标架，使得插入项直接作用于算符上，而该算符本身就插入在 w 标架中。接下来我们利用式 (294) 将上述积分内的表达式按 w 展开为幂级数进行计算。经过简单推导可得

$$\left(\frac{\partial F}{\partial w}\right)^{-1}\frac{\partial F}{\partial u^i}=\alpha_i+\beta_i w+\gamma_i w^2+\mathcal{O}\left(w^3\right), \tag{296}$$

where the expansion coefficients are quickly confirmed to be

其中可以很快验证展开系数为

$$\alpha_i=\frac{1}{a}\frac{\partial y}{\partial u^i}$$

$$\beta_i=\frac{1}{a}\frac{\partial a}{\partial u^i}-\frac{b}{a^2}\frac{\partial y}{\partial u^i} \tag{297}$$

$$\gamma_i=a\frac{\partial}{\partial u^i}\left(\frac{b}{2a^2}\right)+\left(\frac{b^2}{a^3}-\frac{1}{2}\frac{c}{a^2}\right)\frac{\partial y}{\partial u^i}.$$

For the complex conjugate factor in the antighost insertion, the result is the same, with all quantities complex conjugated. This means that with the oscillator expansion $b(w)=\sum_n b_n/w^{n+2}$ and the analogous one for $\bar{b}(\bar{w})$, we have

对于反鬼插入中的复共轭因子，结果是一致的，仅需将所有量取复共轭。这意味着结合振子展开式 $b(w)=\sum_n b_n/w^{n+2}$ 和 $\bar{b}(\bar{w})$ 的对应展开式，我们得到

$$\mathcal{B}\left[\frac{\partial}{\partial u^i}\right]=-\left(\alpha_i b_{-1}+\beta_i b_0+\gamma_i b_1+\overline{\alpha}_i\bar{b}_{-1}+\overline{\beta}_i\bar{b}_0+\overline{\gamma}_i\bar{b}_1+\cdots\right),$$

(298)

where the dots indicate terms with oscillators $b_n,\bar{b}_n$ with $n\geq 2$, which are not needed for the dilaton computation. Now we find from (284)

其中省略号代表带有振子 $b_n,\bar{b}_n$ 和 $n\geq 2$ 的项，这些项在 dilaton 计算中不需要。现在我们由式 (284) 得到

$$\Omega_2(D)=\left(-\frac{1}{2\pi i}\right)du^1\wedge du^2\mathcal{B}\left[\frac{\partial}{\partial u^1}\right]\mathcal{B}\left[\frac{\partial}{\partial u^2}\right](c_1c_{-1}-\bar{c}_1\bar{c}_{-1})\,|0\rangle \tag{299}$$

$$=\left(\frac{1}{2\pi i}\right)du^1\wedge du^2\left[\alpha_1\gamma_2-\alpha_2\gamma_1-(\overline{\alpha}_1\overline{\gamma}_2-\overline{\alpha}_2\overline{\gamma}_1)\right]|0\rangle,$$

where we note that within the brackets, the second term, in parentheses, is the complex conjugate (c.c.) of the first. Moreover, both terms are antisymmetric in the 1 and 2 labels. Using the values of the α and γ coefficients, we find

其中我们注意到，括号内第二个带括号的项是第一个项的复共轭 (c.c.)。此外，两项关于标号 1 和标号 2 都是反对称的。代入 α 和 γ 系数的值，我们得到

$$\Omega_2(D)=\left(\frac{1}{2\pi i}\right)du^1\wedge du^2\left[\frac{\partial y}{\partial u^1}\frac{\partial}{\partial u^2}\left(\frac{b}{2a^2}\right)-\frac{\partial y}{\partial u^2}\frac{\partial}{\partial u^1}\left(\frac{b}{2a^2}\right)-(\text{ c.c. })\right]|0\rangle$$

(300)

where terms involving the product $\frac{\partial y}{\partial u^1}\frac{\partial y}{\partial u^2}$ cancel out for the $1\leftrightarrow 2$ antisymmetry. We then find

其中由于 $1\leftrightarrow 2$ 的反对称性，包含乘积 $\frac{\partial y}{\partial u^1}\frac{\partial y}{\partial u^2}$ 的项相互抵消。随后我们得到

$$\Omega_2(D)=\frac{1}{2\pi i}\left[dy\wedge d\left(\frac{b}{2a^2}\right)-d\bar{y}\wedge d\left(\frac{\bar{b}}{2\bar{a}^2}\right)\right]|0\rangle \tag{301}$$

which is simply

其结果简单为

$$\Omega_2(D)=\frac{1}{2\pi i}dy\wedge d\bar{y}\left[\frac{\partial}{\partial\bar{y}}\left(\frac{b}{2a^2}\right)+\frac{\partial}{\partial y}\left(\frac{\bar{b}}{2\bar{a}^2}\right)\right]. \tag{302}$$

This is our final result for the integrated dilaton insertion two-form. Since the vacuum state corresponds to the identity operator, we deleted $|0\rangle$, and effectively this form integrates a function over the surface.

这就是积分后 dilaton 插入二形式的最终结果。由于真空态对应单位算符，我们删除了 $|0\rangle$，该形式实际上是在曲面上对函数做积分。

For the χ state one-form, we have from (286) and (298)

对于 χ 态一形式，我们由式 (286) 和式 (298) 得到

$$\Omega_1(\chi) = -\frac{1}{2\pi i}\left(-\frac{1}{2}du^1\mathcal{B}\left[\frac{\partial}{\partial u^1}\right] - \frac{1}{2}du^2\mathcal{B}\left[\frac{\partial}{\partial u^2}\right]\right)(c_0 - \bar{c}_0)|0\rangle \tag{303}$$

$$= -\frac{1}{2\pi i}\left(\frac{1}{2}du^1\left(\beta_1 - \overline{\beta}_1\right) + \frac{1}{2}du^2\left(\beta_2 - \overline{\beta}_2\right)\right)|0\rangle.$$

Using the values of the β_i , one readily finds

代入 β_i 的值，很容易得到

$$\Omega_1(\chi) = \frac{1}{2\pi i}\left[-\frac{1}{2}d\left(\ln\frac{a}{\bar{a}}\right) + \frac{b}{2a^2}dy - \frac{\bar{b}}{2\bar{a}^2}d\bar{y}\right]. \tag{304}$$

This is our final result for the integrated χ one-form. One quickly verifies that $\Omega_2(D) = -\mathrm{d}\Omega_1(\chi)$.

这就是积分后 χ 一形式的最终结果。可以快速验证得到 $\Omega_2(D) = -\mathrm{d}\Omega_1(\chi)$ 。

Note that in the analysis above, we have ignored the possibility of the point y meeting the contours associated with $\mathcal{B}\left[\partial/\partial v^i\right]$ insertions. It has been shown in [88] (section 6.1) that there is no extra contribution from these regions of y integration.

注意在上述分析中，我们忽略了 y 点与 $\mathcal{B}\left[\partial/\partial v^i\right]$ 插入对应的围道相交的可能性。文献 [88](第 6.1 节) 已经证明，在这些 y 积分区域内不会产生额外贡献。

At this point, to make these forms compute geometrically recognizable quantities, we now place a condition on an a priori arbitrary conformal metric ρ on the surface. A conformal metric is one where the length element is $ds = \rho\,|dz|$. We demand that the real and imaginary parts of w function as Riemann normal coordinates at the location of the puncture [91]. To deal with the dilaton, it suffices to demand the lowest-order version of the constraint: that the metric ρ^w in the w frame satisfy $\partial_w \rho^w|_{w=0} = \partial_{\bar{w}}\rho^w|_{w=0} = 0$. That is, we are setting the linear parts of the w dependence of the metric to zero. To implement this, we write the metric as $ds = \rho^w\,|dw|$ in the w coordinate, with

至此，为了让这些形式计算出几何上可辨识的量，我们现在对曲面上先验任意的共形度量 ρ 给出一个条件。共形度量的线元为 $ds = \rho\,|dz|$ 。我们要求 w 的实部和虚部在穿刺位置充当黎曼法坐标 [91]。要处理 dilaton，只需要要求该约束的最低阶形式: 即 w 标架下的度量 ρ^w 满足 $\partial_w \rho^w|_{w=0} = \partial_{\bar{w}}\rho^w|_{w=0} = 0$ 。也就是说，我们将度量对 w 依赖的线性项都设为零。为实现这一点，我们在 w 坐标系下将度量写为 $ds = \rho^w\,|dw|$ ，其中

$$\rho^w(w) = \rho\left|\frac{dz}{dw}\right| = \left(\rho(y) + (z-y)\left.\frac{\partial\rho}{\partial z}\right|_y + (\bar{z}-\bar{y})\left.\frac{\partial\rho}{\partial\bar{z}}\right|_y + \cdots\right)|a + bw + \cdots|$$

(305)

where dots represent terms of higher order in w that will not be relevant. Noting that at the puncture, where the derivatives are evaluated, z can be traded for y , and to linear order in $w, \bar{w}$, we have

其中省略号代表 w 的高阶项，这些项无关紧要。注意到在导数取值的穿刺处，z 可以替换为 y，且在 $w,\bar{w}$ 的线性阶下，我们有

$$\begin{aligned}\rho^{w}(w)=\rho\left|\frac{dz}{dw}\right|&=\left(\rho(y)+aw\frac{\partial\rho}{\partial y}\Big|_{y}+\bar{a}\bar{w}\frac{\partial\rho}{\partial\bar{y}}\Big|_{y}+\cdots\right)|a|\\&\times\left(1+\frac{b}{2a}w+\frac{\bar{b}}{2\bar{a}}\bar{w}+\cdots\right)\\&=\rho(y)|a|\left[1+aw\left(\frac{\partial}{\partial y}\log\rho+\frac{b}{2a^{2}}\right)+\bar{a}\bar{w}\left(\frac{\partial}{\partial\bar{y}}\log\rho+\frac{\bar{b}}{2\bar{a}^{2}}\right)+\cdots\right].\end{aligned}$$

(306)

The condition that first derivatives of ρ^{w} vanish at y requires the linear terms in w in the above expression to vanish. This sets

ρ^{w} 的一阶导数在 y 处为零的条件要求上述表达式中 w 的线性项为零，由此得到

$$\frac{b}{2a^{2}}=-\frac{\partial}{\partial y}\log\rho,\ \frac{\bar{b}}{2\bar{a}^{2}}=-\frac{\partial}{\partial\bar{y}}\log\rho. \tag{307}$$

With these relations, the dilaton and χ forms become, respectively,

利用这些关系，dilaton 和 χ 形式分别变为

(308)

$$\Omega_{2}(D)=\left(\frac{1}{2\pi i}\right)dy\wedge d\bar{y}\left[-2\frac{\partial}{\partial\bar{y}}\frac{\partial}{\partial y}\log\rho\right],$$

$$\Omega_{1}(\chi)=\frac{1}{2\pi i}(-i)\left[d\theta_{a}-idy\frac{\partial}{\partial y}\log\rho+id\bar{y}\frac{\partial}{\partial\bar{y}}\log\rho\right],$$

where we set $a=|a|\,e^{i\theta_{a}}$, so that θ_{a} is the phase of the a coefficient ($|a|$ is the so-called mapping radius of the local coordinate). Comparing with the curvature two-form and the geodesic curvatures in (289), we have

此处我们设 $a=|a|\,e^{i\theta_{a}}$ ，因此 θ_{a} 是 a 系数的相位 ($|a|$ 是局部坐标的所谓映射半径)。与 (289) 式中的曲率二形式和测地曲率对比，我们得到

$$\Omega_{2}(D)=-\frac{1}{2\pi}K^{(2)}\text{ and }\Omega_{1}(\chi)=-\frac{1}{2\pi}k^{(1)}. \tag{309}$$

This result confirms the anticipated role of the dilaton D and χ insertions. We integrate the dilaton form over the surface minus the coordinate disks of the punctures, where the other external states are inserted. This prevents the collision of the dilaton and the external states. Having stopped at the boundaries, we must include the integration of the χ one-form over the circles bounding the coordinate disks. Both integrals together give the Euler number of the surface, independent of the fiducial metric ρ used to create the family of local coordinates.

该结果证实了胀子插入项 D 和 χ 的预期作用。我们对胀子形式在减去打孔坐标圆盘后的曲面上积分，其余外部态都插入在这些打孔处。这避免了胀子与外部态发生碰撞。在积分到边界停止后，我们必须补充 χ 一形式在坐标圆盘的边界圆周上的积分。两项积分共同给出曲面的欧拉数，与构造局部坐标族所用的参考度规 ρ 无关。

Note that the proof of (309) is valid for arbitrary off-shell states $A_1, \cdots, A_n$ in the correlation function. This establishes that for a fixed Riemann surface, integrating $\Omega^{(g,n+1)}_{6g-6+2n+2}(D, A_1, \cdots, A_n)$ over the surface minus the coordinate disks, with moduli describing the location where D is inserted, supplemented by the integral of $\Omega^{(g,n+1)}_{6g-6+2n+1}(\chi, A_1, \cdots, A_n)$ over the boundaries of the coordinate disks, with moduli describing the location where χ is inserted, gives [88]

注意 (309) 式的证明对关联函数中任意离壳态 $A_1, \cdots, A_n$ 都成立。由此可得: 对于固定黎曼曲面，在减去坐标圆盘的曲面上对 $\Omega^{(g,n+1)}_{6g-6+2n+2}(D, A_1, \cdots, A_n)$ 积分 (其中模参数描述 D 的插入位置)，再补充在坐标圆盘边界上对 $\Omega^{(g,n+1)}_{6g-6+2n+1}(\chi, A_1, \cdots, A_n)$ 积分 (其中模参数描述 χ 的插入位置)，结果为 [88]

$$(2 - 2g - n)\, \Omega^{(g,n)}_{6g-6+2n}(A_1, \cdots, A_n)\,. \tag{310}$$

This is the result needed for a dilaton shift to change the coupling constant of the theory, for (minus) the Euler number is the power of the coupling constant appearing in the string interactions in the combination $(g_s)^{-\chi_{g,n}}\{A_1, \cdots, A_n\}_g$ (see Eq. (107)).

这正是胀子平移改变理论耦合常数所需的结果，因为负欧拉数就是弦相互作用中耦合常数在组合 $(g_s)^{-\chi_{g,n}}\{A_1, \cdots, A_n\}_g$ 里的幂次 (见式 (107))。

The proof of the dilaton theorem in string field theory uses the above result as a guide to construct the field redefinition that maps the combination of the string field action and the measure with one value of the string coupling, to the same combination with a small variation of the coupling constant. To leading order, there is a shift of the string field by the dilaton $|D\rangle$. This is the first term of a not-quite-legal gauge transformation with gauge parameter $|\chi\rangle$, and the next term in the field redefinition involves the insertion of $|\chi\rangle$ on the three-string vertex. But higher-order terms in the redefinition need modifications, because the illegality of χ begins to matter. The redefinition is also different from the one in the background independence analysis, because the dilaton is not a primary. The full redefinition is given in [88], Section 8, in the language of a ”Hamiltonian” that induces the field redefinition via the antibracket.

弦场论中胀子定理的证明以上述结果为指导，构造了场重定义: 将给定弦耦合取值下弦场作用量与测度的组合，映射到弦耦合发生微小变化后的同一组合。领头阶下，弦场发生了由胀子 $|D\rangle$ 带来的平移。这是一个不严格合法的规范变换的第一项，其规范参数为 $|\chi\rangle$ ，场重定义的下一项涉及在三弦顶点插入 $|\chi\rangle$ 。但重定义的高阶项需要修正，因为 χ 的不合法性开始产生影响。该重定义也不同于背景独立性分析中的重定义，因为胀子不是本原场。完整的重定义可见文献 [88] 第 8 节，它用“哈密顿量”的语言给出，该哈密顿量通过反括号诱导出场重定义。

Algebraic Structures Underlying String Field Theories

弦场论的基础代数结构

In string field theory, as presently formulated, the string field action S is a functional of a string field. The string field is defined as a general vector in the state space of a suitably chosen conformal or super-conformal two-dimensional field theory. Most versions of string field theories share a common algebraic structure. For classical open string field theories, the structure on the $\mathcal{H}_o$ space where the string field lives is a cyclic A_∞ algebra of multilinear string field products satisfying a set of relations and endowed with a cyclic inner product [56]. For classical closed string field theories, the structure on $\mathcal{H}_c$ is an L_∞ algebra of multilinear string field products satisfying a set of relations and a symmetric inner product [6]. Such kind of L_∞ structures also exist for ordinary field theories, as reviewed in detail and elaborated in [92].

在目前表述的弦场论中，弦场作用量 S 是弦场的泛函。弦场被定义为经适当选取的共形或超共形二维场论状态空间中的一个一般矢量。大多数弦场论版本共享共同的代数结构。对于经典开弦场论，弦场所处的 $\mathcal{H}_o$ 空间上的结构是一个循环 A_∞ 代数，它由满足一组关系的多线性弦场乘积构成，并配有循环内积 [56]。对于经典闭弦场论，$\mathcal{H}_c$ 上的结构是一个 L_∞ 代数，它由满足一组关系的多线性弦场乘积构成，并配有对称内积 [6]。这类 L_∞ 结构也存在于普通场论中，文献 [92] 对此做了详细综述与拓展。

In this section, we will describe such structures and show how they can be used to construct actions that satisfy the Batalin-Vilkovisky master equation and are thus are guaranteed a consistent quantization. The free string field theories require the definition of the lowest product in the algebra, a product with one input string field, giving an output string field. This is simply a linear operator, identified with the BRST operator of the conformal field theory. The free string field theory also requires the inner product, which also arises naturally from the conformal field theory. The interaction terms in the string field theory require the definition of string vertices, which we will discuss in the following section. For the time being, we simply assume suitable vertices exist that allow one to construct the multilinear string products needed for the A_∞ and L_∞ algebras.

本节我们将描述这类结构，并说明如何利用它们构造满足巴塔林-维尔可夫斯基主方程的作用量，从而保证该作用量具有自洽量子化。自由弦场论要求定义代数中最低阶的乘积，即仅含一个输入弦场、输出一个弦场的乘积。这本质上是一个线性算符，对应共形场论的 BRST 算符。自由弦场论还需要内积，它同样自然来源于共形场论。弦场论的相互作用项要求定义弦顶点，我们将在下一节讨论。眼下我们直接假设存在合适的顶点，可供我们构造 A_∞ 和 L_∞ 代数所需的多线性弦乘积。

For the quantum closed string field theory, the L_∞ structure is modified. While the multilinear products of the classical theory arise from string amplitudes on genus zero punctured Riemann surfaces, the multilinear products of the quantum theory require punctured Riemann surfaces of genus one and higher. Consistency relations satisfied by the products are also modified for the quantum theory, some authors calling the resulting structure a "quantum L_∞ algebra." For classical open string field theories, a tractable quantum theory requires including closed strings as part of the spectrum. The resulting open-closed string field theories have an algebraic structure in which A_∞ and L_∞ subalgebras are extended to a larger self-consistent structure involving interactions of both open and closed strings.

对于量子闭弦场论，L_∞ 结构会发生修改。经典理论的多线性乘积来源于零亏格带孔黎曼曲面上的弦振幅，而量子理论的多线性乘积需要亏格为一及更高的带孔黎曼曲面。量子理论中乘积满足的相容性关系也发生了修改，部分作者将得到的结构称为“量子 L_∞ 代数”。对于经典开弦场论，一个可处理的量子理论需要将闭弦纳入谱中。由此得到的开-闭弦场论具有如下代数结构: A_∞ 和 L_∞ 子代数被扩展为更大的自洽结构，同时包含开弦和闭弦的相互作用。

As discussed in section ”Bosonic and Superstring Field Theories”, for open superstrings, heterotic strings, and type II strings, a complete construction in the L_∞ framework involves adding an extra copy of the string field, a copy that is needed to get the action and equations of motion to work out but that turns out to describe free, decoupled degrees of freedom. There exist some versions of superstring field theories that do not fit the A_∞ or L_∞ structures but are rather based on Wess-Zumino-Witten (WZW)-like algebraic structures; these will be reviewed in section ”Superstring Field Theories in the Large Hilbert Space”.

正如“玻色弦与超弦场论”一节所讨论的，对于开超弦、杂化弦和 II 型弦，L_∞ 框架下的完整构造需要额外添加一份弦场的拷贝，这份拷贝是推导作用量和运动方程所必需的，但最终描述的是自由的、退耦的自由度。存在部分版本的超弦场论不满足 A_∞ 或 L_∞ 结构，而是基于类似 Wess-Zumino-Witten(WZW) 的代数结构；我们将在“大希尔伯特空间中的超弦场论”一节对其进行综述。

Homotopy algebras are a useful organizing principle in string field theory, and their development was strongly influenced by the string constructions they had to describe. There have also been a number of recent applications of homotopy algebras to ordinary field theory, such as W algebras, double copy relations, and double field theory [93-97]. Some recent work on homotopy algebras and BV quantization in the context of superstring field theory can be found in [74,98].

同伦代数是弦场论中一种有用的组织原理，它的发展也深受它需要描述的弦构造的影响。近年来同伦代数也被大量应用于普通场论，例如 W 代数、双拷贝关系和双重场论 [93-97]。关于超弦场论背景下同伦代数与 BV 量子化的近期研究可参见文献 [74,98]。

A_∞ Algebras and Classical Open String Field Theory

A_∞ 代数与经典开弦场论

In this section, we shall give a formal description of the A_∞ algebra that underlies the formulation of tree-level open string field theory. In the following section, we shall discuss the relation between this formal structure and the formulation of open string field theory described in section ”Tree-Level Open String Field Theory”.

在本节中，我们将正式描述构建树级开弦场论基础的 A_∞ 代数。在下一节，我们会讨论该形式结构与“树级开弦场论”一节中介绍的开弦场论构造之间的关系。

For this algebraic structure, we simply assume that we work with a vector space V with elements with a natural $\mathbb{Z}$ grading, which gives the ”degree” of the elements. For most aspects, all that matters is the degree modulo two; we have even elements if the degree is zero (mod 2) or odd elements if the degree is one (mod

2). We shall denote by d_A the degree of the element A . In the analysis in this section, even degree elements will behave as Grassmann even and odd degree elements will behave as Grassmann odd. [13]

对于这个代数结构，我们简单假设我们在矢量空间 V 上工作，该空间的元素带有自然的 $\mathbb{Z}$ 分次，用来给出元素的“次数”。对于大部分讨论而言，重要的只是次数模二的结果: 次数模二为零时是偶元素，模二为一时是奇元素。我们将用 d_A 表示元素 A 的次数。在本节的分析中，偶次元素的行为对应格拉斯曼偶，奇次元素的行为对应格拉斯曼奇。[13]

We now define the products of the A_∞ algebra. For this, we define a set of multilinear maps $b_n : V^{\otimes n} \to V$ with $n = 1, 2, \ldots$. These are products, as they take n vectors in V as input, and the output is a vector in V . All products are declared to be of degree minus one, meaning that

我们现在定义 A_∞ 代数的乘积。为此，我们定义一组带有 $n = 1, 2, \ldots$ 的多线性映射 $b_n : V^{\otimes n} \to V$ 。这些映射就是乘积，因为它们将 n 个 V 中的矢量作为输入，输出仍是 V 中的一个矢量。规定所有乘积的次数均为负一，这意味着:

$$\deg\left(b_n\left(A_1, \cdots, A_n\right)\right) = -1 + \sum_{i=1}^{n} d_{A_i}. \tag{311}$$

In order to formulate the consistency conditions satisfied by these multilinear maps, one forms the larger vector space called the ”tensor co-algebra” $T(V)$

为了表述这些多线性映射满足的相容性条件，我们构造一个更大的矢量空间，称为“张量余代数” $T(V)$

$$T(V) \equiv V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots \tag{312}$$

Acting on such a space, one can consider the linear operator $\mathbf{b}$ of degree minus one, which essentially is the sum of all multilinear maps. When it acts on the $V^{\otimes n}$ subspace of $T(V)$, the operator $\mathbf{b}$ is defined as follows:

作用在这个空间上，我们可以考虑次数为负一的线性算子 $\mathbf{b}$ ，它本质上就是所有多线性映射的和。当它作用在 $T(V)$ 的 $V^{\otimes n}$ 子空间上时，算子 $\mathbf{b}$ 的定义如下:

$$\mathbf{b} = \sum_{i=1}^{n} \sum_{j=0}^{n-i} \mathbb{1}^{\otimes j} \otimes b_i \otimes \mathbb{1}^{n-i-j}, \text{ on } V^{\otimes n}. \tag{313}$$

As one can see, in here, the product b_i acts on all the possible length- i list of consecutive entries in $V^{\otimes n}$. It is possible to unpack this, and to break $\mathbf{b}$ into a sum of linear operators:

可以看到，乘积 b_i 作用在 $V^{\otimes n}$ 中所有可能长度为 i 的连续元素序列上。我们可以将其展开，把 $\mathbf{b}$ 拆分为若干线性算子的和:

$$\mathbf{b} = \sum_{i=1}^{\infty} \mathbf{b}_i = \mathbf{b}_1 + \mathbf{b}_2 + \cdots . \tag{314}$$

Then, consistent with (313), we have

那么，与式 (313) 一致，我们有

$$\mathbf{b}_i = \sum_{j=0}^{n-i} \mathbb{1}^{\otimes j} \otimes b_i \otimes \mathbb{1}^{n-i-j}, \text{ on } V^{\otimes n}, \text{ for } i \leq n. \tag{315}$$

When $i > n, \mathbf{b}_i$ acting on $V^{\otimes n}$ is zero. Note that $\mathbf{b}_i$ is an operator on $T(V)$, while b_i is an operator on $V^{\otimes i}$. In fact, an operator like $\mathbf{b}_i$ acting in this fashion on $T(V)$ is called a coderivation, and a sum of coderivations, such as $\mathbf{b}$, is also a coderivation. [14] In fact, the most general coderivation can be specified in terms of multilinear products [99], as we did for $\mathbf{b}$, which is defined by the collection of b_i ' s.

当 $i > n, \mathbf{b}_i$ 作用在 $V^{\otimes n}$ 上结果为零。注意 $\mathbf{b}_i$ 是作用在 $T(V)$ 上的算子，而 b_i 是作用在 $V^{\otimes i}$ 上的算子。事实上，像 $\mathbf{b}_i$ 这样作用在 $T(V)$ 上的算子称为余导子，而诸如 $\mathbf{b}$ 这样多个余导子的和仍然是余导子。[14] 实际上，最一般的余导子可以像我们对 $\mathbf{b}$ 做的那样，通过多线性乘积 [99] 来确定，而 $\mathbf{b}$ 本身就是由全体 b_i 所定义的。

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[13] When we apply this to open string field theory, however, this assignment of degree is opposite to that of the Grassmanality of the vertex operator, e.g., a vertex operator carrying odd ghost number will correspond to an even degree element of the algebra. To avoid confusion, we have introduced the symbol d_A for degree of A , which is to be distinguished from the Grassmanality of the vertex operators that will be denoted by $(-1)^A$. The relation between the formalism developed in this section and that used in section "Tree-Level Open String Field Theory" will be explained in section "Relation Between Different Classical Open SFT Formalisms".

[13] 然而，当我们将这应用于开弦场论时，这种度数定义与顶点算子的格拉斯曼奇偶性相反，例如，携带奇数鬼数的顶点算子对应代数中的一个偶度数元素。为避免混淆，我们引入符号 d_A 表示 A 的度数，它与顶点算子的格拉斯曼奇偶性不同，后者将用 $(-1)^A$ 表示。本节建立的形式体系与" 树级开弦场论" 一节所用形式体系的关系，将在" 不同经典开弦场论形式体系之间的关系" 一节中说明。

Acting on $V^{\otimes 3}$, for example, $\mathbf{b}$ becomes

例如，作用在 $V^{\otimes 3}$ 上时，$\mathbf{b}$ 变为

$$\mathbf{b} = b_1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes b_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes b_1 + b_2 \otimes \mathbb{1}$$

$$+ \mathbb{1} \otimes b_2 + b_3 \text{ , on } V^{\otimes 3} \text{ ,} \tag{316}$$

and acting on $A \otimes B \otimes C \in V^{\otimes 3}$, we would get

而作用在 $A \otimes B \otimes C \in V^{\otimes 3}$ 上时，我们得到

$$\mathbf{b}(A \otimes B \otimes C)$$

$$= b_1(A) \otimes B \otimes C + (-1)^{d_A} A \otimes b_1(B) \otimes C + (-1)^{d_A+d_B} A \otimes B \otimes b_1(C)$$

$$+ b_2(A,B) \otimes C + (-1)^{d_A} A \otimes b_2(B,C) + b_3(A,B,C),$$

(317)

with the sign factors arising when odd degree b ' s go across the states. Note that the result on the right-hand side is an element of $V \oplus (V \otimes V) \oplus (V \otimes V \otimes V)$.

符号因子产生于奇次数的 b 穿过态时。注意，右侧的结果是 $V \oplus (V \otimes V) \oplus (V \otimes V \otimes V)$ 中的一个元素。

The (graded) commutator of coderivations is in fact a coderivation, so $\mathbf{b}^2 = \frac{1}{2}\{\mathbf{b}, \mathbf{b}\}$ is a coderivation. The consistency condition on the products is simply the requirement that the coderivation $\mathbf{b}^2$ vanishes, so that $\mathbf{b}$ has the property of a differential:

余导子的 (分次) 对易子本身仍是余导子，因此 $\mathbf{b}^2 = \frac{1}{2}\{\mathbf{b}, \mathbf{b}\}$ 是一个余导子。乘积的一致性条件简单来说就是要求余导子 $\mathbf{b}^2$ 为零，从而使 $\mathbf{b}$ 满足微分的性质:

$$\mathbf{b}^2 = 0\,. \tag{318}$$

The content of these conditions is fully seen by consideration of the action on the various summands of $T(V)$, namely, $V^{\otimes n}$ with $n \geq 1$. On $V^{\otimes n}$, the condition can be expressed as

通过考察这些条件作用在 $T(V)$ 的各加项上，即作用在带 $n \geq 1$ 的 $V^{\otimes n}$ 上，就能完全看清这些条件的内容。在 $V^{\otimes n}$ 上，条件可表示为

$$\sum_{i=1}^{n} \mathbf{b}_i \mathbf{b}_{n+1-i} = 0, \text{ on } V^{\otimes n} \tag{319}$$

The product here is composition of the operator action. The A_∞ algebra is simply the vector space V with the products b_n , which, when assembled into $\mathbf{b}$ as in (314), satisfy (318).

此处的乘积是算符作用的复合。A_∞ 代数就是带有乘积 b_n 的向量空间 V ，这些乘积按 (314) 的方式组装为 $\mathbf{b}$ 后满足 (318) 式。

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[14] To fully define a coderivation, one must define a coproduct $\overline{\Delta} : T(V) \to T(V) \otimes' T(V)$, with the prime to distinguish this product from just the tensor product within $T(V)$. The coproduct is a linear operator acting as follows:

[14] 要完全定义一个余导子，必须先定义余乘 $\overline{\Delta} : T(V) \to T(V) \otimes' T(V)$ ，这里加撇是为了将它与 $T(V)$ 内部的张量乘积区分开。余乘是线性算符，作用形式如下:

$$\overline{\Delta}(A_1 \otimes \cdots \otimes A_n) = \sum_{k=1}^{n-1} (A_1 \otimes \cdots A_k) \otimes' (A_{k+1} \cdots A_n), n \geq 2,\ \overline{\Delta}(A_1) = 0.$$

A linear operator $\mathbf{a}$ on $T(V)$ is a coderivation if $\overline{\Delta}\mathbf{a} = \left(\mathbf{a}\otimes'\mathbf{I} + \mathbf{I}\otimes'\mathbf{a}\right)\overline{\Delta}$, with $\mathbf{I}$ the identity operator on $T(V)$.

$T(V)$ 上的线性算符 $\mathbf{a}$ 满足 $\overline{\Delta}\mathbf{a} = \left(\mathbf{a}\otimes'\mathbf{I} + \mathbf{I}\otimes'\mathbf{a}\right)\overline{\Delta}$ 时是余导子，其中 $\mathbf{I}$ 是 $T(V)$ 上的恒等算符。

The first few cases of the conditions are

条件的前几种情况为

$$\text{Acting on } V: 0 = \mathbf{b}_1\mathbf{b}_1 ,$$

$$\text{Acting on } V^{\otimes 2}: 0 = \mathbf{b}_1\mathbf{b}_2 + \mathbf{b}_2\mathbf{b}_1 , \tag{320}$$

$$\text{Acting on } V^{\otimes 3}: 0 = \mathbf{b}_1\mathbf{b}_3 + \mathbf{b}_2\mathbf{b}_2 + \mathbf{b}_3\mathbf{b}_1 .$$

More explicitly, using the string products and the symbol $\circ$ for composition,

更明确地说，使用弦乘积和符号 $\circ$ 表示复合,

$$\text{Acting on } V: 0 = b_1 \circ b_1 ,$$

$$\text{Acting on } V^{\otimes 2}: 0 = b_1 \circ b_2 + b_2 \circ (b_1 \otimes \mathbb{1} + \mathbb{1} \otimes b_1) , \tag{321}$$

$$\text{Acting on } V^{\otimes 3}: 0 = b_1 \circ b_3 + b_2 \circ (b_2 \otimes \mathbb{1} + \mathbb{1} \otimes b_2) ,$$

$$+b_3 (b_1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes b_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes b_1).$$

We will write $b_1(A) = QA$, with Q the BRST operator. The product b_2 of the algebra can be written as $b_2(A,B) = AB$. The b_3 product is the first of an infinite series of "homotopies" $(b_3, b_4, b_5, ...)$. It is written as $b_3(A,B,C) = (A,B,C) \in V$. The three identities above then give, when acting on states,

我们记为 $b_1(A) = QA$ ，其中 Q 是 BRST 算符。代数的乘积 b_2 可写为 $b_2(A,B) = AB$ 。乘积 b_3 是无穷级数“同伦”$(b_3, b_4, b_5, ...)$ 中的第一项，记为 $b_3(A,B,C) = (A,B,C) \in V$ 。作用在态上时，上述三个恒等式给出

$$0 = Q^2 A$$

$$0 = Q(AB) + (QA)B + (-1)^{d_A} A(QB), \tag{322}$$

$$0 = Q(A,B,C) + (AB)C + (-1)^{d_A} A(BC),$$

$$+ (QA,B,C) + (-1)^{d_A}(A,QB,C) + (-1)^{d_A+d_B}(A,B,QC).$$

The first identity is the nilpotency of the BRST operator. The second identity states that Q is an odd derivation of the product. The third identity shows that strict associativity of the product, that is, having $(AB)C = -(-1)^{d_A} A(BC)$, is not required. Here, associativity holds "up to homotopy," that is, up to terms involving the homotopy b_3 and the BRST operator. In the general framework of classical open string field theory, we do not require strict associativity, and we have nontrivial products $b_{n\geq 3}$. Of course, if we have an associative product b_2, all the higher homotopies vanish, and the algebraic structure of the string field theory is simpler, with Q and b_2 being the only ingredients. This is the case for the cubic classical open string field theory formulated by Witten [5].

第一个恒等式是 BRST 算符的幂零性。第二个恒等式表明 Q 是乘积的奇导子。第三个恒等式说明不要求乘积满足严格结合律，即不要求 $(AB)C = -(-1)^{d_A} A(BC)$ 成立。在此，结合律“到同伦”成立，也就是到包含同伦 b_3 和 BRST 算符的项成立。在经典开弦场论的一般框架中，我们不要求严格结合律，并且存在非平凡乘积 $b_{n\geq 3}$。当然，如果我们有一个结合乘积 b_2，所有高阶同伦都为零，弦场论的代数结构会更简单，仅包含 Q 和 b_2。这就是威滕提出的三次经典开弦场论 [5] 的情况。

We shall now introduce a bilinear inner product on the vector space of the algebra. This is a structure that, given two elements of V, gives us a complex number. We thus have a bilinear inner product $V \otimes V \to \mathbb{C}$ written as $(A,B) \in \mathbb{C}$, for $A,B \in V$. This structure is required to have the following exchange property

我们现在将在该代数的向量空间上引入一个双线性内积。给定 V 的两个元素，这个结构会给出一个复数。这样我们就得到了一个双线性内积 $V \otimes V \to \mathbb{C}$，对 $A,B \in V$ 记作 $(A,B) \in \mathbb{C}$。该结构需要满足以下交换性质

$$(A,B) = -(-1)^{d_A d_B}(B,A). \tag{323}$$

The bilinear form is symplectic in that it is antisymmetric when both vectors are of even degree. It is only symmetric when both vectors are of odd degree. The BRST operator Q satisfies the folding over property

该双线性形式是辛形式，即当两个向量都是偶次数时它是反对称的，只有当两个向量都是奇次数时它才是对称的。BRST 算符 Q 满足折卷性质

$$(QA,B) = -(-1)^{d_A}(A,QB). \tag{324}$$

The inner product implements the cyclicity property of the products. Intuitively, cyclicity is the natural result of open string vertex operators being located on the boundary of a disk. In such a situation, the ordering of the operators along the boundary matters, but cyclicity reflects the absence of a special "first" operator on the boundary. The axioms require that the bilinear form and the products satisfy

内积实现了乘积的循环性质。直观来看，循环性是开弦顶点算子位于圆盘边界上的自然结果。在这种情况下，算子沿边界的排序是重要的，而循环性反映了边界上不存在特殊的" 第一个" 算子。公理要求双线性形式与乘积满足

$$(A_1, b_n(A_2, \cdots, A_{n+1})) = (-1)^{\#}(A_2, b_n(A_3, \cdots, A_{n+1}, A_1)), \tag{325}$$

with # the sign factor necessary to rearrange the states into the final ordering. Since b_n is odd and degree operates like Grassmanality, we have

其中 # 是将态重新排列为最终排序所需的符号因子。由于 b_n 是奇次数的，且次数的性质和格拉斯曼性一致，因此我们有

$$\# = d_{A_2} + d_{A_1}\left(1 + d_{A_2} + d_{A_3} + \cdots + d_{A_{n+1}}\right). \tag{326}$$

An A_∞ algebra equipped with an inner product satisfying the properties above is called a cyclic A_∞ algebra. This concludes the presentation of the algebraic structure and its axioms.

一个配备了满足上述性质的内积的 A_∞ 代数被称为循环 A_∞ 代数。至此我们已经介绍完了该代数结构及其公理。

For the application to open string field theory, we now note that the general version of this theory requires a Grassmann algebra $\mathcal{G}$ because the target-space theory contains anticommuting fields, such as ghosts. For classical bosonic open string field theory, complex numbers suffice, since the spacetime fields are valued on the complex numbers. As explained at the beginning of section "Bosonic and Superstring Field Theories", the space V that for open strings would naturally be the vector space $\mathcal{H}_o$ of the BCFT must be upgraded to a $\mathcal{G}$ module in the quantum theory, which, with a bit of abuse of notation, we still denote by $\mathcal{H}_o$. The degree of an element is now the sum of the degrees of the BCFT basis vector and the degree of the associated target space field, given by its Grassmanality. The string field products thus have inputs that are each elements of the module $\mathcal{H}_o$ and an output that is also an element of the module $\mathcal{H}_o$. The inner product is then a map from the tensor product of the modules to the Grassmann algebra: $\mathcal{H}_o \otimes \mathcal{H}_o \to \mathcal{G}$.

对于开弦场论的应用，我们现在指出，该理论的一般形式需要格拉斯曼代数 $\mathcal{G}$ ，因为靶空间理论包含反对易场，例如鬼场。对于经典玻色开弦场论，复数就足够了，因为时空场是复数值的。正如节" 玻色弦和超弦场论" 开头所解释的，开弦对应的空间 V 自然是 BCFT 的向量空间 $\mathcal{H}_o$ ，在量子理论中必须升级为 $\mathcal{G}$ 模，稍作 notation 滥用后，我们仍将其记作 $\mathcal{H}_o$ 。一个元素的次数现在是 BCFT 基矢的次数加上关联靶场的次数 (由其格拉斯曼性给出)。因此弦场乘积的输入都是模 $\mathcal{H}_o$ 的元素，输出也仍是模 $\mathcal{H}_o$ 的元素。此时内积是一个从模的张量积到格拉斯曼代数的映射: $\mathcal{H}_o \otimes \mathcal{H}_o \to \mathcal{G}$ 。

The open string field Φ contains Grassmann odd open string vertex operators multiplied by even target space fields and Grassmann even vertex operators times odd target space fields—thus a Grassmann odd object overall. In this A_∞ picture, however, this string field must be viewed as an even degree object in the algebra: $d_\Phi = 0(\bmod 2)$. The classical master action $S(\Phi)$ for the string field is simply a sum over the various products of the A_∞ algebra. We claim that

开弦场 Φ 包含格拉斯曼奇开弦顶点算子乘偶靶场，以及格拉斯曼偶顶点算子乘奇靶场——因此整体是格拉斯曼奇对象。但在这个 A_∞ 框架下，该弦场必须被视为代数中的偶次数对象: $d_\Phi = 0 (\mod 2)$ 。弦场的经典主作用量 $S(\Phi)$ 就是 A_∞ 代数各类乘积的和。我们指出

$$S(\Phi) = \sum_{n=1}^{\infty} \frac{1}{n+1} (\Phi, b_n (\Phi, \cdots, \Phi)), \tag{327}$$

is the master action and satisfies the classical master equation. We will show this in the next section by relating the above action to the previously obtained, consistent action (139).

该式就是主作用量，且满足经典主方程。我们将在下一节通过把上述作用量与之前得到的自洽作用量 (139) 联系起来证明这一点。

One surprising feature of the results given in this subsection is the apparent mismatch between the Grassmanality of a vertex operators and its degree, which is taken to be even (odd) for vertex operators carrying odd (even) ghost number. This is visible, for example, in (322), where every time we move Q through an element A of the algebra, we pick a factor of $(-1)^{d_A}$, which is 1 for odd ghost number operators and -1 for even ghost number operators. This can be contrasted with (142) where every time we move Q through a Grassmann odd vertex operator inside the product $[\cdots]$, we pick up a minus sign. Let us first note that the rules of moving Grassmann odd elements through the vertex operators inside a product could differ if we changed the definition of the product. In (142), we were implicitly assuming that the product $[\cdots]$ followed the prescription given in section "Signs of Forms in $\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ ": inside the correlators, we arrange all the vertex operators first, followed by $\mathcal{B}$ and boundary state insertions to the right. However, we could have arranged the vertex operators interspersed with $\mathcal{B}$ insertions, in which case every time we move Q from one Grassmann odd vertex operator to the next one, Q also passes through a $\mathcal{B}$, and as a result, we do not pick up any sign. A description that could lead to the rules (322) is provided by regarding the products $b_n (A_1, \cdots, A_n)$ as the result of contraction of a surface state with the ket states $|A_1\rangle, \cdots |A_n\rangle$. As discussed at the beginning of section "Tree-Level Open String Field Theory", the ket state $|A\rangle$ has opposite Grassmanality to that of the vertex operator A , and hence an odd vertex operator will naturally lead to even ket states. In section "Relation Between Different Classical Open SFT Formalisms", we shall give the explicit relation between the product $[A_1, \cdots, A_n]$ introduced in section "Tree-Level Open String Field Theory" and the product $b_n (A_1, \cdots, A_n)$ introduced here for all n .

本小节结果的一个惊人特点是，顶点算符的格拉斯曼奇偶性与其度数明显不匹配——对于携带奇数(偶数) 鬼数的顶点算符，其度数被取为偶 (奇)。这一点例如在 (322) 中清晰可见: 每次我们将 Q 穿过代数中的元素 A ，都会得到一个因子 $(-1)^{d_A}$ ，该因子对奇数鬼数算符为 1，对偶数鬼数算符为-1。这与 (142) 形成对比，在 (142) 中，每次我们将 Q 穿过乘积 $[\cdots]$ 中的格拉斯曼奇顶点算符，都会得到一个负号。首先我们要注意: 如果我们改变乘积的定义，移动格拉斯曼奇元素穿过乘积内顶点算符的符号规则也会不同。在 (142) 中，我们默认乘积 $[\cdots]$ 遵循“$\widehat{\mathcal{P}}_{g,b,n_c,n_o}$ 中形式的符号”一节给出的规定: 在关联函数内部，我们先排列所有顶点算符，再将 $\mathcal{B}$ 和边界态插入放在顶点算符右侧。然而，我们也可以将顶点算符与 $\mathcal{B}$ 穿插排列，这种情况下，每次我们将 Q 从一个格拉斯曼奇顶点算符移动到下一个，Q 也会穿过一个 $\mathcal{B}$ ，因此我们不会得到任何额外符号。能够导出规则 (322) 的一种描述是: 将乘积 $b_n(A_1,\cdots,A_n)$ 视为一个曲面态与右矢态 $|A_1\rangle,\cdots|A_n\rangle$ 缩并的结果。正如“树级开弦场论”一节开头所讨论的，右矢态 $|A\rangle$ 与顶点算符 A 的格拉斯曼奇偶性相反，因此奇数顶点算符自然对应偶右矢态。在“不同经典开弦场论形式体系之间的关系”一节中，我们将给出“树级开弦场论”一节引入的乘积 $[A_1,\cdots,A_n]$ 和本文此处对所有 n 引入的乘积 $b_n(A_1,\cdots,A_n)$ 之间的明确关系。

Relation Between Different Classical Open SFT Formalisms

不同经典开弦场论形式体系之间的关系

In this section, we shall discuss the relation between the various quantities defined in section ”Tree-Level Open String Field Theory” and those introduced in section ” A_∞ Algebras and Classical Open String Field Theory”. Both deal with classical open string theory.

在本节中，我们将讨论“树级开弦场论”一节中定义的各类量，与“A_∞ 代数与经典开弦场论”一节中引入的量之间的关系。二者均研究经典开弦理论。

We begin with the definition of the bilinear inner product. We had the BPZ product $\langle\cdot,\cdot\rangle$ used in section ”Tree-Level Open String Field Theory” and the inner product $(\cdot,\cdot)$ used in the algebraic analysis of section ” A_∞ Algebras and Classical Open String Field Theory”. These two, we claim, are related as follows:

我们从双线性内积的定义开始。“树级开弦场论”一节中使用了 BPZ 乘积 $\langle\cdot,\cdot\rangle$ ，而“A_∞ 代数与经典开弦场论”一节的代数分析中使用了内积 $(\cdot,\cdot)$ 。我们指出，这二者满足如下关系:

$$(A,B)=(-1)^{A+1}\langle A,B\rangle'. \tag{328}$$

Here A in the exponent is the (mod 2) ghost number of A , which coincides with the Grassmanality of A (A is a vertex operator or a string field vertex operator).

此处指数中的 A 是 A 的 (模 2) 鬼数，它与 A 的格拉斯曼性一致 (A 是顶点算子或弦场顶点算子)。

To relate this to the degree of A that appeared in section ” A_∞ Algebras and Classical Open String Field Theory”, we shall make the identification [15]

为了将它和“A_∞ 代数与经典开弦场论”一节中出现的 A 的次数联系起来，我们做如下标识 [15]

$$A = d_A + 1\,(\bmod\ 2)\,. \tag{329}$$

Note that we are assigning opposite Grassmanality to A than what was assigned in the previous subsection. As a consistency check, the exchange property (323) for $(\cdot,\cdot)$ follows immediately from the exchange property of the BPZ inner product

注意，我们赋予 A 的格拉斯曼性与上一小节赋予的相反。作为一致性检验，$(\cdot,\cdot)$ 的交换性质 (323) 可直接由 BPZ 内积的交换性质得到

$$\langle B, A\rangle' = (-1)^{AB}\langle A, B\rangle', \tag{330}$$

the relation between the inner products, and the above Grassmanality/degree relation. Indeed,

即内积之间的关系，以及上述格拉斯曼性/次数关系。事实上，

$$(B, A) = (-1)^{B+1}\langle B, A\rangle' = (-1)^{AB+B+1}\langle A, B\rangle' = (-1)^{AB+A+B}\,(A, B)$$

$$= (-1)^{d_A d_B + 1}\,(A, B)\ . \tag{331}$$

Next we relate the star product $\star$ to the A_∞ product b_2 . We have

接下来我们将星乘积 $\star$ 与 A_∞ 乘积 b_2 联系起来。我们有

$$A \star B = (-1)^{A+1} b_2\,(A, B) = (-1)^{A+1}\,(AB)\,, \tag{332}$$

using the definition $(AB) = b_2\,(A, B)$. This gives, using the second relation in (322),

利用定义 $(AB) = b_2\,(A, B)$ 。结合 (322) 中的第二个关系，这给出

$$Q\,(A \star B) = (-1)^{A+1} Q\,(AB) = (-1)^{A}\,((QA)\,B) - (A\,(QB))$$

$$= (QA) \star B + (-1)^{A} A \star QB. \tag{333}$$

More generally, we relate the product $[A_1 \cdots A_n]$ to $b_n\,(A_1, \cdots, A_n)$ via

更一般地，我们将乘积 $[A_1 \cdots A_n]$ 通过下式与 $b_n\,(A_1, \cdots, A_n)$ 联系起来

$$[A_1 \cdots A_n] = (-1)^{n(n+1)/2} \sum_{\sigma} (-1)^{s_\sigma} (-1)^{A_{\sigma(1)} + 2A_{\sigma(2)} + \cdots + nA_{\sigma(n)}}$$

$$\times b_n\left(A_{\sigma(1)}, A_{\sigma(2)}, \cdots A_{\sigma(n)}\right). \tag{334}$$

Here σ is a permutation of $1, \cdots n$, and $(-1)^{s(\sigma)}$ is the sign factor that we pick up when we rearrange the order of the A_i ’ s with the rule $A_i A_j = (-1)^{A_i A_j + 1} A_j A_i$. [16] Using the A_∞ relations (319), we can now easily verify the main identity (142) for the $[\cdots]$ products, for the case of Grassmann odd (i.e., even degree) A_i' ’ s. For the case $n=2$, the above identity leads to

> 此处 σ 是 $1,\cdots n$ 的一个置换，$(-1)^{s(\sigma)}$ 是我们按照规则 $A_iA_j=(-1)^{A_iA_j+1}A_jA_i\cdot$ [16] 重新排列 A_i 顺序时得到的符号因子。利用 A_∞ 关系 (319)，我们现在可以轻松验证 $[\cdots]$ 乘积的主恒等式 (142) 对于格拉斯曼奇 (即偶次数) A_i' 的情形成立。对于 $n=2$ 的情形，上述恒等式给出

$$[A_1,A_2]=A_1\star A_2-(-1)^{A_1A_2}A_2\star A_1. \tag{335}$$

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[15] In mathematical literature, this kind of relation is called a ”suspension.” It relates descriptions of the theory in which the effective Grassmanality of all elements is altered.

> [15] 在数学文献中，这种关系被称为“纬垂”(suspension)。它关联了改变所有元素有效格拉斯曼性的理论描述。

[16] This rule can be derived by examining the left-hand side, defined in (143), whose symmetry follows on account of (141).

> [16] 该规则可以通过研究 (143) 中定义的左手边推导得到，其对称性由 (141) 保证。

Note that we could drop the $(-1)^{n(n+1)/2}$ factor in (334) while replacing $A_{\sigma(i)}$ by $d_{A_{\sigma(i)}}=A_{\sigma(i)}+1$. It also follows from (334) that when all the A_i ’ s are Grassmann odd, i.e., of even degree, we get

> 注意，我们可以在去掉 (334) 中 $(-1)^{n(n+1)/2}$ 因子的同时将 $A_{\sigma(i)}$ 替换为 $d_{A_{\sigma(i)}}=A_{\sigma(i)}+1$ 。由 (334) 还可推出，当所有 A_i 都是格拉斯曼奇，即偶次数时，我们得到

$$[A_1\cdots A_n]=\sum_\sigma b_n\left(A_{\sigma(1)},A_{\sigma(2)},\cdots A_{\sigma(n)}\right). \tag{336}$$

Note that the $(-1)^{s_\sigma}$ factor disappears since for odd A_i ’ s we have $A_iA_j=A_jA_i$ inside [].

> 注意 $(-1)^{s_\sigma}$ 因子会消失，因为对于奇 A_i ，我们在 [] 内有 $A_iA_j=A_jA_i$ 。

Equation (334) provides an implicit definition of $b_n(A_1,\cdots,A_n)$ in terms of world-sheet correlation functions. For this note, we have already defined $[A_1\cdots A_n]$ in terms of the world-sheet correlation functions in section ”Tree-Level Open String Field Theory”. There we summed over all cyclic permutations of the vertex operators to define $\{A_0\cdots A_n\}$ and then used it to define $[A_1\cdots A_n]$ via (143). To define $b_n(A_1,\cdots,A_n)$, we proceed as follows:

> 式 (334) 根据世界面关联函数给出了 $b_n(A_1,\cdots,A_n)$ 的隐式定义。就本文而言，我们已经在“树级开弦场论”一节中根据世界面关联函数定义了 $[A_1\cdots A_n]$ 。我们曾在那里对顶点算符的所有循环排列求和以定义 $\{A_0\cdots A_n\}$ ，再通过式 (143) 用它定义 $[A_1\cdots A_n]$ 。我们按如下步骤定义 $b_n(A_1,\cdots,A_n)$:

- Define $\{A_0A_1\cdots A_n\}^C$ to be that part of $\{A_0A_1\cdots A_n\}$ where the vertex operators are arranged in the cyclic order $0,1,\cdots,n$.

> - 将 $\{A_0A_1\cdots A_n\}^C$ 定义为 $\{A_0A_1\cdots A_n\}$ 中顶点算符按循环序 $0,1,\cdots,n$ 排列的部分。

- Define $[A_1\cdots A_n]^C$ via $\left\langle A_0,[A_1,\cdots,A_n]^C\right\rangle' = \{A_0,A_1,\cdots,A_n\}^C$.

> - 通过 $\left\langle A_0,[A_1,\cdots,A_n]^C\right\rangle' = \{A_0,A_1,\cdots,A_n\}^C$ 定义 $[A_1\cdots A_n]^C$ 。

- Use (334) with $[A_1\cdots A_n]$ replaced by $[A_1\cdots A_n]^C$ on the left-hand side, and pick only the identity permutation in the sum over σ on the right-hand side.

> - 使用式 (334)，将其左侧的 $[A_1\cdots A_n]$ 替换为 $[A_1\cdots A_n]^C$ ，并在右侧对 σ 的求和中仅选取恒等排列。

We now argue for the consistency of the A_∞ master action (327). It follows immediately from the relation between inner products (328) and the relation between products (336) that the A_∞ action coincides with the action (139). Furthermore, since the main identity (142) follows as a consequence of the A_∞ algebra, it is clear that the rest of the results discussed in section ”Tree-Level Open String Field Theory” follow. In particular, the action satisfies the classical BV master equation.

> 现在我们论证 A_∞ 主作用量 (327) 的自洽性。由内积关系式 (328) 和乘积关系式 (336) 可直接得到，A_∞ 作用量与作用量 (139) 一致。此外，由于主恒等式 (142) 是 A_∞ 代数的导出结果，“树级开弦场论”一节中讨论的其余结果显然都成立。特别地，该作用量满足经典 BV 主方程。

L$_\infty$ Algebras and Classical Closed String Field Theory

> ## L$_\infty$ 代数与经典闭弦场论

We now turn to a discussion of the algebraic structure behind classical closed string field theory. While for open strings the relevant algebraic structures arise from homotopy associative algebras, for closed strings, the relevant structures are homotopy Lie algebras. For classical closed string field theory, we have an L_∞ algebra. The L_∞ algebra contains an infinite number of products, all of which are graded symmetric.

> 我们现在来讨论经典闭弦场论背后的代数结构。开弦对应的相关代数结构来源于同伦结合代数，而闭弦对应的相关结构是同伦李代数。对于经典闭弦场论，我们有一个 L_∞ 代数。L_∞ 代数包含无穷多个乘积，所有乘积都是分次对称的。

To work in generality, we assume we have a graded vector space W with elements with a natural $\mathbb{Z}$ grading. We write $B_1,B_2,\cdots \in W$ for elements of fixed degree. We denote the degree of B by d_B . The products are viewed as brackets, such as the Lie algebra bracket with two entries. In the situation used for the conventional formulation of string field theory, there are products with one input, two inputs, and all numbers of higher inputs: [17]

为了进行一般化讨论，我们假设存在一个分次向量空间 W ，其元素具有自然的 $\mathbb{Z}$ 分次。我们将 $B_1, B_2, \cdots \in W$ 记作固定次数的元素，用 d_B 表示 B 的次数。这些乘积可以看作括号，就像包含两个元素的李代数括号。在弦场论的传统表述中，存在一元、二元以及任意更高元输入的乘积: [17]

$$b_1(B_1), b_2(B_1, B_2), b_3(B_1, B_2, B_3), \cdots, \tag{337}$$

etc. Unlike the case of open string field theory, here, the relation to the structures introduced in section "Closed Bosonic String Field Theory" is more straightforward:

等等。与开弦场论的情况不同，本文中它和“闭玻色弦场论”一节引入的结构的关系更直接:

$$b_1(B_1) = QB_1, b_1(B_1, B_2) = [B_1, B_2]_0, b_3(B_1, B_2, B_3) = [B_1, B_2, B_3]_0\,,$$

(338)

where Q is a linear operator in W , to be later identified as the BRST operator of the CFT. The subscript 0 on the brackets [ ] indicates that these will be identified with the genus zero products defined in section "Closed Bosonic String Field Theory". The products b_n are graded commutative. When exchanging any two inputs in a product, we just get a sign factor corresponding to the exchange of the two objects according to their degree

其中 Q 是 W 上的线性算子，后面会被识别为共形场论的 BRST 算子。括号 [ ] 的下标 0 表示它们对应“闭玻色弦场论”一节定义的亏格零乘积。乘积 b_n 是分次交换的。交换乘积中任意两个输入，我们只会得到一个符号因子，该因子对应根据两个对象的次数对它们进行交换的结果

$$b_n(B_1, \cdots, B_k, B_{k+1}, \cdots, B_n) = (-1)^{d_{B_k} d_{B_{k+1}}} b_n(B_1, \cdots, B_{k+1}, B_k, \cdots, B_n)\,.$$

(339)

The products are defined to be of intrinsic degree minus one, so that

按定义，乘积的内蕴次数为减一，因此

$$\deg(b(B_1, \cdots, B_n)) = -1 + \sum_{i=1}^{n} d_{B_i}, \tag{340}$$

in accordance with the comment below (109).

这与式 (109) 下方的说明一致。

In analogy to what we did for the A_∞ algebra, we construct a larger space by adding symmetrized products of the vector space W :

类似我们对 A_∞ 代数做的处理，我们通过添加向量空间 W 的对称化乘积构造一个更大的空间:

$$T(W) = \sum_{n=1}^{\infty} SW^{\otimes n} \tag{341}$$

The summands in this expression are spaces $SW^{\otimes n}$, with S for symmetrized; their elements are written as

该表达式中的加项是空间 $SW^{\otimes n}$ ，其中 S 对应对称化；它们的元素写为

$$B_1 \wedge B_2 \wedge \cdots \wedge B_n \tag{342}$$

where the "wedge" simply means that the order of the B 's can be interchanged with sign factors according to their statistics, just as in the products above. Thus, for example, $B_1 \wedge B_2 = (-1)^{d_{B_1} d_{B_2}} B_2 \wedge B_1$. For a general permutation $\sigma = \{\sigma(1), \cdots, \sigma(n)\}$ of the integers $\{1, \cdots, n\}$, we define

其中“楔积”仅表示 B 的顺序可以根据它们的统计性质交换并得到对应符号因子，和上述乘积中的规则一致。因此举个例子，$B_1 \wedge B_2 = (-1)^{d_{B_1} d_{B_2}} B_2 \wedge B_1$ 。对于整数 $\{1, \cdots, n\}$ 的任意置换 $\sigma = \{\sigma(1), \cdots, \sigma(n)\}$ ，我们定义

$$B_1 \wedge \cdots \wedge B_n = \varepsilon(\sigma; B) B_{\sigma(1)} \wedge \cdots \wedge B_{\sigma(n)}, \tag{343}$$

---

[17] A product b_0 with no input is just a special state in W . If b_0 is included in the set of products, we have a "curved" L_∞ algebra that can play a role in the formulation of string field theory around backgrounds where the classical theory has terms linear in the field [6, 100] .

零输入的 [17] A 乘积 b_0 只是 W 中的一个特殊态。如果把 b_0 纳入乘积集合，我们就得到了“弯曲” L_∞ 代数，它可用于构建围绕经典理论含场 [6, 100] 线性项背景的弦场论表述。

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where $\varepsilon(\sigma; B)$ is the so-called Koszul sign factor, clearly dependent on the permutation σ and the degree of the B entries.

其中 $\varepsilon(\sigma; B)$ 是所谓的 Koszul 符号因子，显然依赖于置换 σ 和 B 个输入的次数。

In order to state the main identity satisfied by the product it is convenient to define the splitting of the set of integers $\{1, \ldots, n\}$ into a first group $\{i_1, \ldots, i_l\}$ of integers and a second group $\{j_1, \ldots, j_k\}$ of integers, clearly with $l + k = n$. We ask that $l \geq 1$ but $k \geq 0$. A splitting has a sign factor $\varepsilon(\sigma)$, where the permutation is that which turns the ordered set into $\{i_1, \ldots, i_l, j_1, \ldots, j_k\}$. Two splittings are said to be equivalent if their first groups contain the same integers, regardless of their order.

为了表述乘积满足的主恒等式，方便起见我们将整数集合 $\{1, \ldots, n\}$ 拆分为第一整数组 $\{i_1, \ldots, i_l\}$ 和第二整数组 $\{j_1, \ldots, j_k\}$ ，显然有 $l + k = n$ 。我们要求 $l \geq 1$ ，但 $k \geq 0$ 。拆分对应一个符号因子 $\varepsilon(\sigma)$ ，其中置换是将原有序集变为 $\{i_1, \ldots, i_l, j_1, \ldots, j_k\}$ 的置换。若两个拆分的第一组包含的整数相同，则二者等价，与整数顺序无关。

We are now able to write the identities satisfied by the products. For this we define an odd coderivation **b** acting on $T(W)$ as follows: [18]

现在我们可以写出乘积满足的恒等式了。为此我们定义一个作用在 $T(W)$ 上的奇余导子 $\mathbf{b}$，定义如下: [18]

$$\mathbf{b}(B_1 \wedge \cdots \wedge B_n) = \sum_{l=1}^{n} \sum_{\sigma'} \varepsilon(\sigma', B) \left(b_l \left(B_{i_1}, \cdots, B_{i_l}\right) \wedge B_{j_1} \wedge \cdots \wedge B_{j_{n-l}}\right), \tag{344}$$

where the sum over σ' denotes the sum over all inequivalent splittings of $\{1, \cdots, n\}$ into a group with l integers and a group with $n-l$ integers (the prime is to remind the reader that this is a restricted sum). Note that a product is used to collapse the first set of states in the splitting into a single state. Thus, for example, we have

其中对 σ' 的求和表示对所有不等价拆分求和，即将 $\{1, \cdots, n\}$ 拆分为一个含 l 个整数的组和一个含 $n-l$ 个整数的组 (撇号提醒读者这是受限求和)。注意拆分后，我们会用一个乘积将第一组的所有态合并为单个态。举例来说，我们有

$$\mathbf{b}(B_1) = b_1(B_1), \tag{345}$$

since the only allowed splitting of $\{1\}$ is into the sets $\{1\}, \{\varnothing\}$. For the case of two inputs, we first note that there are three splittings of $\{1,2\}$: into $\{1\},\{2\}$, into $\{2\},\{1\}$, and into $\{1,2\},\{\varnothing\}$. We therefore have, including the signs,

因为 $\{1\}$ 仅有的允许拆分是集合 $\{1\}, \{\varnothing\}$。对双输入的情况，我们首先注意 $\{1,2\}$ 共有三种拆分: 分为 $\{1\},\{2\}$、分为 $\{2\},\{1\}$，以及分为 $\{1,2\},\{\varnothing\}$。因此计入符号后我们得到:

$$\mathbf{b}(B_1 \wedge B_2) = b_1(B_1) \wedge B_2 + (-1)^{d_{B_1} d_{B_2}} b_1(B_2) \wedge B_1 + b_1(B_1, B_2)$$

(346)

$$= b_1(B_1) \wedge B_2 + (-1)^{d_{B_1}} B_1 \wedge b_1(B_2) + b_1(B_1, B_2).$$

The identity satisfied by the products is simply the condition that the operator $\mathbf{b}$ squares to zero

乘积满足的恒等式就是算子 $\mathbf{b}$ 平方为零的条件

$$\mathbf{b}^2 = 0. \tag{347}$$

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[18] The definition here is given in a form slightly different from the original one in [6], but it is exactly equivalent to it, including all signs.

[18] 此处给出的定义形式与文献 [6] 中的原始定义略有不同，但二者完全等价，包括所有符号都一致。

This can be shown to be equivalent to the nilpotence of the BRST operator Q and the main identities (113) with the last term set to zero. We shall illustrate this with some examples.

可以证明这等价于 BRST 算子 Q 幂零，且主恒等式 (113) 的末项为零。我们会通过几个例子对此进行说明。

For the action on B_1 , as in (345), we find the expected condition of nilpotency of Q :

对于作用在 B_1 上的情况，如式 (345) 所示，我们得到预期的 Q 幂零条件:

$$\mathbf{b}^2 (B_1) = \mathbf{bb} (B_1) = \mathbf{b} (QB_1) = Q (QB_1) = 0. \tag{348}$$

Acting on $B_1 \wedge B_2$ and using (346), we find

作用在 $B_1 \wedge B_2$ 上并利用 (346)，我们得到

$$\mathbf{b}^2 (B_1 \wedge B_2) = \mathbf{b} (QB_1 \wedge B_2) + (-1)^{d_{B_1}} \mathbf{b} (B_1 \wedge QB_2) + \mathbf{b} (b_2 (B_1, B_2)) = 0. \tag{349}$$

Expanding by using (346) for the first two terms,(345) for the last and $Q^2 = 0$, we get

对前两项用 (346) 展开、末项用 (345) 展开，再结合 $Q^2 = 0$ ，我们得到

$$\mathbf{b}^2 (B_1 \wedge B_2) = \left((-1)^{d_{B_1}+1} QB_1 \wedge QB_2 + b_2 (QB_1, B_2)\right)$$

$$+(-1)^{d_{B_1}} (QB_1 \wedge QB_2 + b_2 (B_1, QB_2)) + Qb_2 (B_1, B_2) = 0.$$

(350)

The two terms in $SW^{\otimes 2}$ cancel out and give no new condition. The nontrivial condition arises from the part in W on the above right-hand side:

$SW^{\otimes 2}$ 中的两项相互抵消，不给出新条件。非平凡条件来自上述等式右侧中 W 的部分:

$$Qb_2 (B_1, B_2) + b_2 (QB_1, B_2) + (-1)^{d_{B_1}} b_2 (B_1, QB_2) = 0. \tag{351}$$

This is in agreement with (113) for $N = 2$. This condition states that Q is a (graded) derivation of the two-product.

这与 $N = 2$ 对应的式 (113) 一致。该条件表明 Q 是二元乘积的 (分次) 导子。

Next, we turn to the identity that arises from $\mathbf{b}^2 (B_1 \wedge B_2 \wedge B_3) = 0$. One gets, from the vanishing of the part in W , the condition

接下来我们来看由 $\mathbf{b}^2 (B_1 \wedge B_2 \wedge B_3) = 0$ 得到的恒等式。由 W 部分为零，我们得到条件

$$0 = Qb_3(B_1, B_2, B_3) + b_3(QB_1, B_2, B_3) + (-1)^{d_{B_1}} b_3(B_1, QB_2, B_3)$$

$$+(-1)^{d_{B_1}+d_{B_2}} b_3(B_1, B_2, QB_3)$$

(352)

$$+b_2(b_2(B_1, B_2), B_3) + (-1)^{d_{B_2}d_{B_3}} b_2(b_2(B_1, B_3), B_2)$$

$$+(-1)^{d_{B_1}(d_{B_2}+d_{B_3})} b_2(b_2(B_2, B_3), B_1).$$

This is in agreement with (113) for $N = 3$. Note the three terms on the last line. If these were set to zero, we would have a graded version of the Jacobi identity for the two-product $b_2(B_1, B_2)$. But in L_∞ , this two-product is not a strict Lie bracket, and therefore, it does not satisfy a Jacobi identity. In the L_∞ algebra, the "Jacobiator," namely, the sum of those three terms in question, is set equal to the failure of the higher product $b_3(B_1, B_2, B_3)$ to be a derivation of Q .

这与 $N = 3$ 的式 (113) 一致。注意最后一行的三项。如果将它们置零, 我们就能得到双乘积 $b_2(B_1, B_2)$ 的分次雅可比恒等式。但在 L_∞ 中，该双乘积不是严格李括号，因此它不满足雅可比恒等式。在 L_∞ 代数中，“雅可比子”(即上述那三项的和) 等于高乘积 $b_3(B_1, B_2, B_3)$ 不成为 Q 导子的偏差。

In general, the action of the second $\mathbf{b}$ in $\mathbf{b}^2(B_1 \wedge \ldots \wedge B_n)$ only gives a new identity on account of the vanishing of the terms in W . Thus, making use of (344), the general conditions on the products arise by replacing the parenthesis by a bracket:

一般而言，由于 W 中项的消失，$\mathbf{b}^2(B_1 \wedge \ldots \wedge B_n)$ 中的第二个 $\mathbf{b}$ 作用只会给出一个新的恒等式。因此，利用式 (344)，对乘积的一般条件可通过将括号替换为方括号得到:

$$0 = \sum_{l=1}^{n} \sum_{\sigma'} \varepsilon(\sigma', B) b_{n-l+1}\left(b_l\left(B_{i_1}, \cdots, B_{i_l}\right), B_{j_1}, \cdots, B_{j_{n-l}}\right). \tag{353}$$

Since we have recovered the identity (113), with the last term discarded and all products at genus zero, the construction of the classical master action and the proof that it satisfies the BV master equation proceeds as in section "Closed Bosonic String Field Theory".

由于我们已经恢复了舍去最后一项、所有乘积都对应亏格零的式 (113)，经典主作用的构造以及它满足 BV 主方程的证明都与“闭玻色弦场论”一节中的流程一致。

Finally, as in the case of A_∞ algebra, we can introduce a bilinear inner product that, given two elements B_1, B_2 of the algebra, gives us an element $(B_1, B_2) \in \mathbb{C}$, obeying the symmetry:

最后，和 A_∞ 代数的情况一样，我们可以引入一个双线性内积: 给定代数中的两个元素 B_1, B_2 ，得到一个元素 $(B_1, B_2) \in \mathbb{C}$ ，它满足对称性:

$$(B_1, B_2) = (-1)^{(d_{B_1}+1)(d_{B_2}+1)} (B_2, B_1), \tag{354}$$

and the folding over property

以及折转性质

$$(QB_1, B_2) = (-1)^{d_{B_1}} (B_1, QB_2)\,. \tag{355}$$

Furthermore, using the inner product to define multilinear maps to the complex numbers:

此外，利用内积定义映到复数域的多重线性映射:

$$\{B_1, \cdots, B_n\}_{L_\infty} \equiv (B_1, b_{n-1}(B_2, \cdots B_n))\,, \tag{356}$$

we also demand the graded commutativity property:

我们还要求分次交换性成立:

$$\{B_1, \cdots, B_k, B_{k+1}, \cdots, B_n\}_{L_\infty} = (-1)^{d_{B_k} d_{B_{k+1}}} \{B_1, \cdots, B_{k+1}, B_k, \cdots, B_n\}_{L_\infty}\,.$$

(357)

For the application to closed string field theory, a few remarks are useful. The natural choice for W is the CFT vector space $\mathcal{H}_c$ upgraded to a $\mathcal{G}$ module (still denoted by $\mathcal{H}_c$) to account for Grasmann even and odd target space fields. As we mentioned before, Q is the BRST operator in $\mathcal{H}_c$, and the products are the genus zero brackets. The inner product $(\cdot,\cdot)$, taking $\mathcal{H}_c \otimes \mathcal{H}_c \to \mathcal{G}$ in fact, coincides with $\langle\cdot,\cdot\rangle$, defined in (15). For closed strings, L_∞ degree and Grassmanality coincide: the full string field Ψ is of even degree, and as a vertex operator multiplied by a target space field, it is uniformly Grassmann even. In terms of these quantities, the action S satisfying the classical BV master equation can be written as

就闭弦场论的应用而言，做一些说明是有必要的。W 的自然选取是共形场论向量空间 $\mathcal{H}_c$，为了适配格拉斯曼偶和奇的目标空间场，将其升级为 $\mathcal{G}$ 模 (仍记作 $\mathcal{H}_c$)。如前所述，Q 是 $\mathcal{H}_c$ 中的 BRST 算符，乘积则是亏格零括号。实际上，取 $\mathcal{H}_c \otimes \mathcal{H}_c \to \mathcal{G}$ 的内积 $(\cdot,\cdot)$ 与式 (15) 中定义的 $\langle\cdot,\cdot\rangle$ 一致。对闭弦而言，L_∞ 次数与格拉斯曼奇偶性一致: 全弦场 Ψ 是偶次数的，作为乘以目标空间场的顶点算符，它整体是格拉斯曼偶的。利用这些量，满足经典 BV 主方程的作用量 S 可以写为

$$S = \frac{1}{2}(\Psi, Q\Psi) + \sum_{n=3}^{\infty} \frac{1}{n!}\{\Psi^n\}_{L_\infty}\,, \tag{358}$$

where the string field Ψ is an even degree element of the L_∞ algebra.

其中弦场 Ψ 是 L_∞ 代数的一个偶次数元素。

From A_∞ to L_∞

从 A_∞ 到 L_∞

In section ”Tree-Level Open String Field Theory”, the classical open string field theory was written in terms of products [...], which, as we saw in section ”Relation Between Different Classical Open SFT Formalisms”, can be also expressed in terms of the A_∞ products. We shall now explain how the action can also be written in terms of L_∞ products that are defined in terms of the A_∞ products.

在“树级开弦场论”一节中，经典开弦场论已经用乘积 [$\cdots$] 的形式写出，正如我们在“不同经典开弦场论形式体系间的关系”一节中所见，它也可以用 A_∞ 乘积表示。下面我们将解释如何用由 A_∞ 乘积定义的 L_∞ 乘积写出作用量。

Let us call b_n the A_∞ products and $\bar{b}_n$ the L_∞ products to be constructed from them. The vector space of states is the same, with elements denoted $A_1, A_2, \cdots$. We have that the first product needs no change:

我们记待由 b_n 构造的产物为 A_∞ 乘积，产物 $\bar{b}_n$ 为 L_∞ 乘积。态的向量空间保持不变，元素记为 $A_1, A_2, \cdots$ 。我们可知第一个乘积无需修改:

$$\bar{b}_1 (A_1) \equiv b_1 (A_1) . \tag{359}$$

Clearly $\bar{b}_1 \bar{b}_1 (A_1) = b_1 b_1 (A_1) = 0$, as required. For the second product, we take

显然符合要求的 $\bar{b}_1 \bar{b}_1 (A_1) = b_1 b_1 (A_1) = 0$ 。对于第二个乘积，我们取

$$\bar{b}_2 (A_1, A_2) \equiv b_2 (A_1, A_2) + (-1)^{d_{A_1} d_{A_2}} b_2 (A_2, A_1) . \tag{360}$$

This satisfies the required exchange symmetry $\bar{b}_2 (A_1, A_2) = (-1)^{d_{A_1} d_{A_2}} \bar{b}_2 (A_2, A_1)$ by construction. A short computation confirms that the desired L_∞ property (see (351))

由构造可知，它满足所需的交换对称性 $\bar{b}_2 (A_1, A_2) = (-1)^{d_{A_1} d_{A_2}} \bar{b}_2 (A_2, A_1)$ 。简短计算可证实，期望的 L_∞ 性质 (见式 (351))

$$\bar{b}_1 \bar{b}_2 (A_1, A_2) + \bar{b}_2 \left(\bar{b}_1 (A_1), A_2 \right) + (-1)^{d_{A_1}} \bar{b}_2 \left(A_1, \bar{b}_1 (A_2) \right) = 0, \tag{361}$$

holds because the exactly same equation holds for b_2 and b_1 (see the second equation in (322)). If the A_∞ algebra is in fact associative, there is no b_3 , and we have

成立，因为完全相同的方程对 b_2 和 b_1 成立 (见式 (322) 中的第二个方程)。若 A_∞ 代数确实是结合的，则不存在 b_3 ，我们得到

$$b_2 (b_2 (A_1, A_2), A_3) + (-1)^{d_{A_1}} b_2 (A_1, b_2 (A_2, A_3)) = 0 \tag{362}$$

(see the third equation in (322)). This property in fact guarantees the identity

(见式 (322) 中的第三个方程)。该性质实际上保证了这个恒等式

$$\bar{b}_2 \left(\bar{b}_2 (A_1, A_2), A_3 \right) + (-1)^{d_{A_2} d_{A_3}} \bar{b}_2 \left(\bar{b}_2 (A_1, A_3), A_2 \right)$$

$$+(-1)^{d_{A_1}(d_{A_2}+d_{A_3})}\bar{b}_2\left(\bar{b}_2(A_2,A_3),A_1\right)=0, \tag{363}$$

that must hold for the L_∞ algebra to be a Lie algebra without higher products (see (352)).

对于 L_∞ 代数成为不含高阶乘积的李代数必须成立 (见式 (352))。

More generally, if the A_∞ algebra has higher products, the higher products of the L_∞ algebra are obtained as follows:

更一般地，若 A_∞ 代数存在高阶乘积，则 L_∞ 代数的高阶乘积可按如下方式得到:

$$\bar{b}_n(A_1,\cdots,A_n)\equiv\sum_\sigma\varepsilon(\sigma,A)\,b_n\left(A_{\sigma(1)},\cdots,A_{\sigma(n)}\right). \tag{364}$$

Here we simply sum over the complete set of permutations of the n inputs, with the Koszul sign computed with the rule $A_iA_j=(-1)^{d_{A_i}d_{A_j}}A_jA_i$. This definition implies that the product $\bar{b}_n$ is graded commutative, as required. We will not present here a proof that the higher L_∞ identities for the $\bar{b}$ products are satisfied on account of the full set of identities for the A_∞ products. Note also that since the open string field has even degree, $\bar{b}_n(\psi_o,\cdots,\psi_o)=n!b_n(\psi_o,\cdots,\psi_o)$.

此处我们直接对 n 输入的所有置换求和，Koszul 符号按规则 $A_iA_j=(-1)^{d_{A_i}d_{A_j}}A_jA_i$ 计算。该定义表明乘积 $\bar{b}_n$ 按要求是分次交换的。鉴于 A_∞ 乘积满足全套恒等式，我们在此不证明 $\bar{b}$ 乘积的高阶 L_∞ 恒等式也成立。还需注意，由于开弦场是偶次数的，故 $\bar{b}_n(\psi_o,\cdots,\psi_o)=n!b_n(\psi_o,\cdots,\psi_o)$ 。

Using (334) and (364), we can find the relation between the L_∞ products $\bar{b}_n$ and the products $[\cdots]$ for the open string fields constructed in (334). This takes the form

利用式 (334) 和 (364)，我们可以得到式 (334) 中构造的开弦场的 L_∞ 乘积 $\bar{b}_n$ 与乘积 $[\cdots]$ 之间的关系，形式如下

$$[A_1\cdots A_n]=(-1)^{d_{A_1}+2d_{A_2}+\cdots+nd_{A_n}}\bar{b}_n(A_1,\cdots,A_n). \tag{365}$$

We can also relate the inner products in the L_∞ and A_∞ algebra as follows. Let+denote the inner product in the L_∞ algebra, as introduced at the end of section " L_∞ Algebras and Classical Closed String Field Theory". We equate this to the BPZ inner product in the state space and not the inner product $(\,,\,)\,of the A_\infty$ algebra-the two being related by (328). Thus, we have

我们还可以按如下方式关联 L_∞ 代数和 A_∞ 代数中的内积。记 + 为 L_∞ 代数中的内积,它是我们在“L_∞ 代数与经典闭弦场论”一节末尾引入的。我们将其等同于态空间中的 BPZ 内积，而非 $(\,,\,)\,of the A_\infty$ 代数中的内积——二者通过式 (328) 关联。因此我们有:

$$\langle A_1,A_2\rangle=\langle A_1,A_2\rangle'=(-1)^{d_{A_1}}(A_1,A_2). \tag{366}$$

The symmetry property (354) and (355) follow as consequences of (323) and (324). Equation (366) gives, from (356),

对称性性质 (354) 和 (355) 是 (323) 和 (324) 的推论。结合 (356)，式 (366) 给出:

$$\{A_0A_1\cdots A_n\}_{L_\infty} = \leftarrow A_0, \bar{b}_n(A_1A_2\cdots A_n) + = (-1)^{d_{A_0}}\left(A_0, \bar{b}_n(A_1A_2\cdots A_n)\right)$$

$$= (-1)^{d_{A_0}}\sum_\sigma \varepsilon(\sigma, A)\left(A_0, b_n\left(A_{\sigma(1)},\cdots, A_{\sigma(n)}\right)\right).$$

(367)

With some work, one can show that the symmetry property (357) of the left-hand side follows from the symmetry properties (325) and (326).

经过一些推导，我们可以证明左侧的对称性 (357) 可由对称性 (325) 和 (326) 导出。

Since the open string field ψ_o has even degree d_{ψ_o} , we can express the action (327) as

由于开弦场 ψ_o 是偶次数 d_{ψ_o} ，我们可以将作用量 (327) 写为

$$S(\psi_o) = \sum_{n=1}^{\infty}\frac{1}{n+1}(\psi_o, b_n(\psi_o^n)) = \frac{1}{2}\langle\psi_o, Q\psi_o\rangle + \sum_{n=2}^{\infty}\frac{1}{(n+1)!}\{\psi_o^{n+1}\}_{L_\infty}.$$

(368)

This agrees structurally with the L_∞ action (358) of closed strings.

这在结构上与闭弦的 L_∞ 作用量 (358) 一致。

Equation (365) can be given the following interpretation. When all the A_i ’ s are Grassmann odd (i.e., of even degree), we have

方程 (365) 可以做如下解释。当所有 A_i 都是格拉斯曼奇 (即偶次数) 时，我们有

$$[A_1\cdots A_n] = \bar{b}_n(A_1,\cdots, A_n). \tag{369}$$

If some of the A_i ’ s are Grassmann even (odd degree), then we multiply them by Grassmann odd c -numbers from the left and apply (369). We can then move all the Grassmann odd c -numbers to the left by picking up a minus sign every time the Grassmann odd c-number passes a Grassmann odd A_i inside $[A_1\cdots A_n]$ and picking up a minus sign every time the Grassmann odd c-number passes an odd degree A_i inside $\bar{b}_n(A_1,\cdots A_n)$. It can be easily checked that this prescription leads to (365). Thus, inside the argument of $\bar{b}_n$, the effective Grassmanality of A_i , defined as whether or not a Grassmann odd c -number passing through A_i generates a minus sign, becomes identical to the degree.

如果部分 A_i 是格拉斯曼偶 (奇次数)，我们只需从左侧给它们乘上格拉斯曼奇 c 数再应用 (369)。之后我们把所有格拉斯曼奇 c 数移到最左侧: 格拉斯曼奇 c 数穿过 $[A_1\cdots A_n]$ 内部的格拉斯曼奇 A_i 时，每穿过一次取一个负号；穿过 $\bar{b}_n(A_1,\cdots A_n)$ 内部的奇次数 A_i 时，每穿过一次也取一个负号。很容易验证这个规则能给出 (365)。因此，在 $\bar{b}_n$ 的自变量内，A_i 的有效格拉斯曼性 (定义为格拉斯曼奇 c 数穿过 A_i 时是否产生负号) 和次数完全一致。

Quantum L_∞ Algebra

量子 L_∞ 代数

Classical closed string field theory must be supplemented by additional terms in order to define the full quantum closed string field theory. The Feynman rules of the classical theory do not produce covers of higher genus moduli spaces. This happens because the classical master action does not solve the quantum master equation. In order to produce an action that solves the quantum master equation, one needs higher genus string products. In the language of L_∞ algebra, the products $b_n(A_1, \cdots, A_n) \in W$ now include contributions from all genus, in general. We now write

经典闭弦场论必须补充额外项才能定义完整的量子闭弦场论。经典理论的费曼规则无法生成高亏格模空间的覆盖，这是因为经典主作用量不满足量子主方程。为了得到满足量子主方程的作用量，需要引入高亏格弦乘积。在 L_∞ 代数的语言中，一般来说乘积 $b_n(A_1, \cdots, A_n) \in W$ 现在已经包含了所有亏格的贡献。我们写下

$$b_n(B_1, \cdots, B_n) = \sum_{g=0}^{\infty} g_s^{2g+n-1} b_n^g(B_1, \cdots, B_n), n = 0, 1, \cdots, \tag{370}$$

where with a little abuse of notation, the b_n ’ s now denote the products with all genus contributions, and the genus zero b_n ’ s of the original L_∞ algebra are now written as b_n^0 . A few things should be noted. We now have a product $b_0()$ without an input, which maps nothing to W . It receives contributions from b_0^g products with $g \geq 1 \left(b_0^0() = 0\right)$. In closed string field theory, the inner product of $b_0^g()$ with a state $B \in \mathcal{H}_c$ is related to the one-point function of B on genus g Riemann surfaces. Furthermore, the product $b_1(A_1)$ is QA_1 at genus zero but gets additional contribution from two-point functions on Riemann surfaces of genus $g \geq 1$.

这里存在少许记号滥用: 如今的 b_n 表示包含所有亏格贡献的乘积，原 L_∞ 代数中亏格为零的 b_n 现在记作 b_n^0 。需要注意几点: 我们现在存在一个零输入乘积 $b_0()$ ，它将空输入映射为 W ，它来自 b_0^g 个带 $g \geq 1 \left(b_0^0() = 0\right)$ 的乘积。在闭弦场论中，$b_0^g()$ 和态 $B \in \mathcal{H}_c$ 的内积与亏格 g 黎曼曲面上 B 的单点函数相关。此外，乘积 $b_1(A_1)$ 在亏格零处等于 QA_1 ，但会从亏格 $g \geq 1$ 黎曼曲面的两点函数得到额外贡献。

The identity that must be satisfied by the complete set of products is a generalization of (353), and it takes the form

完整乘积集必须满足的恒等式是式 (353) 的推广，形式为

$$0 = \sum_{l=1}^{n} \sum_{\sigma'} \varepsilon(\sigma', B) b_{n-l+1}\left(b_l\left(B_{i_1}, \cdots, B_{i_l}\right), B_{j_1}, \cdots, B_{j_{n-l}}\right) \tag{371}$$

$$+\frac{1}{2} b_{n+2}(B_1, \cdots, B_n, \varphi_s, \varphi_r)(\varphi_s^c, \varphi_r^c).$$

This identity is equivalent to (113). It is the identity defining the quantum L_∞ algebra. The term in the second line is of purely quantum origin and was not present in (353). Of course, even the first term is different, as it now includes the higher-genus contributions.

该恒等式等价于式 (113)，它是定义量子 L_∞ 代数的恒等式。第二行的项是纯量子起源的，在式 (353) 中不存在。当然，即使第一项也有所不同，因为它现在包含了高亏格贡献。

The abstract L_∞ algebra introduced in section " L_∞ Algebras and Classical Closed String Field Theory" is thus modified to the quantum L_∞ algebra. The main new ingredients are the following:

因此，章节“L_∞ 代数与经典闭弦场论”中引入的抽象 L_∞ 代数被修改为量子 L_∞ 代数，主要新内容如下:

1. We define $\mathbf{b}$ as in (344) with the sum over l starting at 0 and b_i ' s now including higher genus contributions satisfying the identity (371). 2. We also introduce an operator θ from $T(W) \to T(W)$ such that

1. 我们按式 (344) 定义 $\mathbf{b}$ ，其中对 l 的求和从 0 开始，且 b_i 现在包含满足恒等式 (371) 的高亏格贡献。2. 我们还从 $T(W) \to T(W)$ 引入了一个算符 θ ，满足

$$\theta\left(A_1 \wedge \cdots \wedge A_n\right)=\frac{1}{2} A_1 \wedge \cdots \wedge A_n \wedge \varphi_r \wedge \varphi_s\left(\varphi_r^c, \varphi_s^c\right). \tag{372}$$

The quantum L_∞ algebra identity (371) follows from the vanishing of the following operator acting on $T(W)$

量子 L_∞ 代数恒等式 (371) 可由以下作用在 $T(W)$ 上的算符为零推出

$$\pi_1\left(\mathbf{b}^2+\mathbf{b} \theta\right)=0, \tag{373}$$

where π_1 is the projection from $T(W)$ to W . It is shown by Markl [101] that this condition is equivalent to the conceptually clearer condition

其中 π_1 是从 $T(W)$ 到 W 的投影。Markl 在文献 [101] 中证明，该条件等价于概念上更清晰的条件

$$(\mathbf{b}+\theta)^2=0. \tag{374}$$

Open-Closed String Field Theory

开-闭弦场论

In section "Open-Closed String Field Theory", we described a formulation of open-closed string field theory where the open and closed strings appear on similar footing except for some signs (see, e.g., the $(-1)^{j-1}$ factor in the second line of (163)). The difference in sign arises because the closed string fields are even, while open string fields are odd. However, we have seen in section "From A_∞ to L_∞ " that it is possible to reformulate classical open string field theory using the products $\bar{b}_n\left(A_1, \cdots A_n\right)$ that satisfy the same L_∞ algebra as classical closed strings. This suggests that a similar change in sign can be used to rewrite the open-closed string field theory in terms of a regular quantum L_∞ algebra without any extra sign. To this end, we define L_∞ -type multilinear functions from open-closed functions

在“开-闭弦场论”一节中，我们描述了开-闭弦场论的一种表述: 其中开弦与闭弦除了一些符号差异外，地位是对等的 (例如参见 (163) 第二行中的 $(-1)^{j-1}$ 因子)。符号差异的来源是闭弦场是偶场，而开弦场是奇场。但我们在“从 A_∞ 到 L_∞ ”一节中已经看到，可以用满足与经典闭弦相同 L_∞ 代数的乘积 $\bar{b}_n(A_1,\cdots A_n)$ 重新表述经典开弦场论。这说明我们可以通过类似的符号调整，将开-闭弦场论改写为正则量子 L_∞ 代数的形式，不需要额外的符号。为此，我们由开-闭函数定义 L_∞ 型多线性函数

$$\{A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\}_{L_\infty}$$

$$\equiv(-1)^{d_{A_2^o}+2d_{A_3^o}+\cdots+(n-1)d_{A_n^o}}\{A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\},\tag{375}$$

with $\{A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\}$ as defined in (157). When restricted to classical open string field theory, this expression reduces to $\{A_1^o,\cdots,A_n^o\}_{L_\infty}$ via (365)- (367). Then the open-closed string field theory action defined in terms of $\{\cdots\}_{L_\infty}$ will have the same form as (165) since for Grassmann odd open string states there is no difference between $\{\cdots\}_{L_\infty}$ and $\{\cdots\}$. In terms of the new L_∞ multilinear functions, however, the $(-1)^{j-1}$ factor for open strings will be absent in (163), and we can treat open and closed strings on equivalent footing.

其中 $\{A_1^c,\cdots,A_{n_c}^c;A_1^o,\cdots,A_{n_o}^o\}$ 按 (157) 定义。当限制到经典开弦场论时，该表达式通过 (365)-(367) 约化为 $\{A_1^o,\cdots,A_n^o\}_{L_\infty}$ 。此后用 $\{\cdots\}_{L_\infty}$ 定义的开-闭弦场论作用量将和 (165) 形式一致，因为对格拉斯曼奇开弦态来说，$\{\cdots\}_{L_\infty}$ 和 $\{\cdots\}$ 没有区别。但用新的 L_∞ 多线性函数表示时，(163) 中将不存在开弦的 $(-1)^{j-1}$ 因子，我们就可以对等看待开弦和闭弦。

The open-closed system is a novel and fascinating structure in terms of homotopy algebras. Much of the novelty is present for the bosonic string case. As Kajiura and Stasheff discuss [102, 103], there are various ways of thinking of the algebraic structure of the open-closed string theory. One can view it as a deformation of the A_∞ structure on the open string state space controlled by the L_∞ algebra on the closed string state space. Physicists note that indeed open-closed string field theory shows how to write classical open string field theories on general closed string backgrounds. Mathematically, one can focus on the L_∞ algebra acting on the open string state space as a graded vector space. This is the homotopy version of a Lie algebra L acting on a vector space M , where M is viewed as a representation of L . There has been renewed work in formulating the open-closed string field theory in the language of coderivations, finding interesting limits, and the implications for constant terms in the action [104-106].

开-闭系统从同伦代数的角度看是一种新颖迷人的结构。大部分新颖性在玻色弦情形中就已经体现。如 Kajiura 和 Stasheff 讨论的 [102, 103]，有多种方式理解开-闭弦场论的代数结构。可以将它视为开弦态空间上 A_∞ 结构的形变，该形变由闭弦态空间上的 L_∞ 代数控制。物理学家已经指出，开-闭弦场论确实说明了如何构造一般闭弦背景下的经典开弦场论。从数学角度，我们可以重点关注作用在作为分次向量空间的开弦态空间上的 L_∞ 代数。这就是李代数 L 作用在向量空间 M 上的同伦版本，其中 M 被视为 L 的一个表示。近来已有不少利用余导数语言表述开-闭弦场论、寻找有趣极限以及分析作用量中常数项 implications 的研究 [104-106]。

Homotopy Transfer

同伦传递

We considered the Wilsonian effective action in section "Wilsonian Effective Action". The main takeaway was that starting from a consistent string field theory, one could obtain, via a suitable projection of the string field, a subset of fields for which there is a consistent effective field theory. In algebraic terms and focusing here only on the classical theories, we have the original string field theory with an A_∞ or L_∞ algebra. The procedure to be discussed below, going under the name of homotopy transfer, builds an A_∞ or L_∞ algebra on the subset of fields [70, 71, 102, 103, 107] . These algebras thus define the effective field theory of that subset of fields. The construction clarifies the general conditions that must be satisfied in the selection of the projector down to a subset of fields. It also provides explicit formulae for the terms in the effective action.

我们已经在“威尔逊有效作用量”一节中讨论过威尔逊有效作用量。核心结论是: 从一个自洽的弦场论出发，通过对弦场做合适的投影，可以得到一组子场，这组子场对应一个自洽的有效场论。限于经典理论从代数角度来看，我们原弦场论具有一个 A_∞ 代数或 L_∞ 代数。下文将要讨论的方法叫做同伦传递，它会在子场 [70, 71, 102, 103, 107] 上构造出一个 A_∞ 代数或 L_∞ 代数。这些代数就定义了该子场的有效场论。该构造明确了将场投影到子场的过程中, 选择投影算子需要满足的一般条件, 还给出了有效作用量中各项的显式公式。

Homotopy Transfer for $\mathbf{A}_\infty$. Consider classical open strings formulated in the framework of A_∞ algebras. The A_∞ algebras will be described by coderivations on the tensor algebra $T(V)$ as we discussed in section " A_∞ Algebras and Classical Open String Field Theory". A slight change of notation is helpful here. We wrote before $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 + \cdots$. We will set $\mathbf{M} = \mathbf{b}$; write $\mathbf{Q} = \mathbf{b}_1$, as this is the BRST operator; and define $\mathbf{m} = \mathbf{b}_2 + \mathbf{b}_3 + \cdots$. All in all, we have the degree one operator

$\mathbf{A}_\infty$ 的同伦传递。考虑 A_∞ 代数框架下表述的经典开弦。正如我们在“A_∞ 代数与经典开弦场论”一节中讨论的, A_∞ 代数可以用张量代数 $T(V)$ 上的上导子描述。这里稍微修改一下记号会更方便。我们之前写的是 $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 + \cdots$ 。现在我们设 $\mathbf{M} = \mathbf{b}$ ；记 $\mathbf{Q} = \mathbf{b}_1$ ，因为它就是 BRST 算符；再定义 $\mathbf{m} = \mathbf{b}_2 + \mathbf{b}_3 + \cdots$ 。综上，我们得到一次算符

$$\mathbf{M} = \mathbf{Q} + \mathbf{m}, \ \mathbf{m} = \sum_{n=2}^{\infty} \mathbf{b}_n. \tag{376}$$

The main identity satisfied by the products is $\mathbf{M}^2 = 0$, which implies

乘积满足的核心恒等式是 $\mathbf{M}^2 = 0$ ，由此可得

$$\mathbf{Q}^2 = 0, \ \text{ and } \ \mathbf{Qm} + \mathbf{mQ} + \mathbf{m}^2 = 0. \tag{377}$$

We now follow the discussion of [72]. We introduce a projector P from V to a subspace $\bar{V}$; this is the subspace of degrees of freedom we want to write an effective field theory for

现在我们按照文献 [72] 的讨论展开。我们引入一个从 V 到子空间 $\bar{V}$ 的投影算子 P ；这个子空间就是我们要构造有效场论的自由度所在的空间

$$P : V \to \bar{V},\ P^2 = P. \tag{378}$$

The A_∞ inner product must be consistent with the projector:

A_∞ 内积必须与投影算子相容:

$$(A_1, PA_2) = (PA_1, A_2)\,. \tag{379}$$

We also demand that P and Q commute

我们还要求 P 与 Q 对易

$$QP = PQ. \tag{380}$$

We extend this projector to a degree zero operator $\mathbf{P} : T(V) \to T\left(\bar{V}\right)$ by defining the action

我们将这个投影算子延拓为零次算符 $\mathbf{P} : T(V) \to T\left(\bar{V}\right)$，其作用定义为

$$\mathbf{P}\left(A_1 \otimes A_2 \otimes \cdots \otimes A_n\right) \equiv PA_1 \otimes PA_2 \otimes \cdots \otimes PA_n, \tag{381}$$

so that

因此

$$\mathbf{P}^2 = \mathbf{P}, \tag{382}$$

and

且

$$\mathbf{QP} = \mathbf{PQ}. \tag{383}$$

Associated with the projector, we can introduce a linear operator h of degree 1 (thus odd) in V satisfying the conditions

与投影算子相伴，我们可以在 V 中引入一个满足如下条件的一次线性算符 h (因此是奇算符)

$$hP = Ph = 0,\ h^2 = 0,\ Qh + hQ = 1 - P. \tag{384}$$

If P projects on to a subspace containing all the $L_0 = 0$ states, for example, then we can take $h = \frac{b_0}{L_0}(1 - P)$
.

例如，如果 P 投影到包含所有 $L_0 = 0$ 态的子空间，我们就可以取 $h = \frac{b_0}{L_0}(1 - P)$。

The next definition tells us how h is promoted to an operator $\mathbf{h}$ acting on $T(V)$. In fact, $\mathbf{h}$ maps $V^{\otimes n}$ to itself as follows:

> 接下来的定义说明了如何将 h 提升为作用在 $T(V)$ 上的算符 $\mathbf{h}$ 。实际上，$\mathbf{h}$ 按如下方式将 $V^{\otimes n}$ 映射到自身:

$$h(A_1 \otimes A_2 \otimes \cdots \otimes A_n) \equiv (hA_1) \otimes PA_2 \otimes \cdots \otimes PA_n$$

$$+(-1)^{d_{A_1}} A_1 \otimes hA_2 \otimes \cdots \otimes PA_n$$

$$\vdots \; \vdots$$

$$+(-1)^{d_{A_1}+d_{A_2}+\cdots+d_{A_{n-1}}} A_1 \otimes A_2 \otimes \cdots \otimes hA_n.$$

(385)

The surprising fact of this definition is that projectors are only inserted to the right of the insertion of h . The sign factors simply indicate that h is odd. The definition is such that properties (384) holding on V) get upgraded to identities in $T(V)$:

> 这个定义的一个特殊之处在于，投影算子只插在 h 插入点的右侧。符号因子只是说明 h 是奇的。该定义保证了在 V 上成立的性质 (384) 可以升级为 $T(V)$ 中的恒等式:

$$\mathbf{hP} = \mathbf{Ph} = 0, \; \mathbf{h}^2 = 0, \; \mathbf{Qh} + \mathbf{hQ} = \mathbf{1} - \mathbf{P}. \tag{386}$$

The first three equations are almost obvious; the last one takes more work. It is straightforward to see just by acting on $A_1 \otimes A_2$ that the unusual placement of projectors is needed for the last equation to work out.

> 前三个方程几乎是显然的；最后一个方程需要更多推导。直接作用在 $A_1 \otimes A_2$ 上就能很容易看出，投影算子这种特殊的放置方式是最后一个方程成立的必要条件。

We can now state that we have an A_∞ algebra on $\bar{V}$, although we will describe it as one in V in such a way that any action on the kernel of P is manifestly zero. The homotopy transfer result for A_∞ states that the operator $\overline{\mathbf{M}}$ defining the new products is given by Erbin et al. [71] and Koyama et al. [72]

> 现在我们可以指出，$\bar{V}$ 上存在一个 A_∞ 代数，不过我们会将它描述为 V 中的代数，这样任何对 P 核的作用都明显为零。A_∞ 的同伦传递结果表明，定义新乘积的算符 $\overline{\mathbf{M}}$ 由 Erbin 等人 [71] 和 Koyama 等人 [72] 给出

$$\overline{\mathbf{M}} = \mathbf{PQP} + \mathbf{Pm}\frac{1}{1+\mathbf{hm}}\mathbf{P} \tag{387}$$

and satisfies $\overline{\mathbf{M}}^2 = 0$, as required to define an A_∞ algebra. Additionally, one must show that $\overline{\mathbf{M}}$ is a coderivation in the tensor coalgebra (see, e.g.,[72], Section 8.1.3). Finally, the new string products must be cyclic, just as the original ones. The $\mathbf{P}$ to the right of each term on the right-hand side ensures that $\overline{\mathbf{M}}$ acts nontrivially only on $T(\bar{V})$. The $\mathbf{P}$ to the left of each term shows that the range of $\overline{\mathbf{M}}$ is in $T(\bar{V})$. The fraction is defined by its power series expansion

且满足定义 A_∞ 代数所需的 $\overline{\mathbf{M}}^2 = 0$ 条件。此外，必须证明 $\overline{\mathbf{M}}$ 是张量上代数中的一个余导子 (参见例如文献 [72] 第 8.1.3 节)。最后，新的弦乘积必须和原乘积一样是循环的。右侧每一项右边的 $\mathbf{P}$ 保证 $\overline{\mathbf{M}}$ 仅在 $T(\bar{V})$ 上非平凡作用。每一项左边的 $\mathbf{P}$ 表明 $\overline{\mathbf{M}}$ 的像落在 $T(\bar{V})$ 中。该分式由其幂级数展开定义

$$\frac{1}{1+\mathbf{hm}} = 1 - \mathbf{hm} + \mathbf{hmhmm} - \cdots . \tag{388}$$

Let us verify that $\overline{\mathbf{M}}^2 = 0$. The square of the first term vanishes: $\mathrm{PQPPQP} = \mathbf{PQPQP} = \mathbf{PQQP} = 0$ since $\mathbf{Q}$ and $\mathbf{P}$ commute. Given this, we have

我们来验证 $\overline{\mathbf{M}}^2 = 0$。第一项的平方为零: 由于 $\mathbf{Q}$ 与 $\mathbf{P}$ 对易，$\mathrm{PQPPQP} = \mathbf{PQPQP} = \mathbf{PQQP} = 0$。由此我们得到

$$\overline{\mathbf{M}}^2 = \mathbf{P}\left\{\mathbf{Q}, \mathbf{m}\frac{\mathbf{1}}{\mathbf{1}+\mathbf{hm}}\right\}\mathbf{P} + \mathbf{Pm}\frac{\mathbf{1}}{\mathbf{1}+\mathbf{hm}}\mathbf{Pm}\frac{\mathbf{1}}{\mathbf{1}+\mathbf{hm}}\mathbf{P}. \tag{389}$$

With a little work, the first term is seen to give

稍加计算即可发现，第一项给出

$$\mathbf{P}\left\{\mathbf{Q}, \mathbf{m}\frac{1}{1+\mathbf{hm}}\right\}\mathbf{P}$$

$$= \mathbf{P}\{\mathbf{Q}, \mathbf{m}\}\frac{1}{1+\mathbf{hm}}\mathbf{P} + \mathbf{Pm}\frac{1}{1+\mathbf{hm}}[\mathbf{Q}, 1+\mathbf{hm}]\frac{1}{1+\mathbf{hm}}\mathbf{P}$$

$$= -\mathbf{Pm}^2\frac{\mathbf{1}}{\mathbf{1}+\mathbf{hm}}\mathbf{P} + \mathbf{Pm}\frac{\mathbf{1}}{\mathbf{1}+\mathbf{hm}}\left((\mathbf{1}-\mathbf{P})\,\mathbf{m} + \mathbf{hm}^2\right)\frac{\mathbf{1}}{\mathbf{1}+\mathbf{hm}}\mathbf{P}.$$

Substituting this back into (389), one finds that indeed $\overline{\mathbf{M}}^2 = 0$.

将其代回式 (389)，确实可以得到 $\overline{\mathbf{M}}^2 = 0$。

It is instructive to expand the result (387) to write the new products $\overline{\mathbf{M}} = \overline{\mathbf{Q}} + \overline{\mathbf{m}}_2 + \overline{\mathbf{m}}_3 + \cdots$. One finds:

展开结果 (387) 写出新乘积 $\overline{\mathbf{M}} = \overline{\mathbf{Q}} + \overline{\mathbf{m}}_2 + \overline{\mathbf{m}}_3 + \cdots$ 是有启发意义的。我们得到:

$$\overline{\mathbf{Q}} = \mathbf{PQP}$$

$$\overline{\mathbf{m}}_2 = \mathbf{Pm}_2\mathbf{P}$$

$$\overline{\mathbf{m}}_3 = \mathbf{P}\left[\mathbf{m}_3 - \mathbf{m}_2\mathbf{hm}_2\right]\mathbf{P}, \tag{390}$$

$$\overline{\mathbf{m}}_4 = \mathbf{P}\left[\mathbf{m}_4 - \mathbf{m}_2\mathbf{h}\mathbf{m}_3 - \mathbf{m}_3\mathbf{h}\mathbf{m}_2 + \mathbf{m}_2\mathbf{h}\mathbf{m}_2\mathbf{h}\mathbf{m}_2\right]\mathbf{P}.$$

Acting on string fields A_i , defining $\bar{A}_i = PA_i$, and using $m_2(A \otimes B) = A \star B$, we have for the first three products the explicit action:

作用在弦场 A_i 上，定义 $\bar{A}_i = PA_i$ 并利用 $m_2(A \otimes B) = A \star B$ ，我们得到前三个乘积的显式作用:

$$\bar{Q}A_1 = PQ\bar{A}_1$$

$$\bar{m}_2(A_1 \otimes A_2) = P\left(\bar{A}_1 \star \bar{A}_2\right),$$

$$\bar{m}_3(A_1 \otimes A_2 \otimes A_3) = Pm_3\left(\bar{A}_1,\bar{A}_2,\bar{A}_3\right) - P\left(h\left(\bar{A}_1 \star \bar{A}_2\right) \star \bar{A}_3 + \bar{A}_1 \star h\left(\bar{A}_2 \star \bar{A}_3\right)\right).$$

(391)

We can connect to the results of section ”Wilsonian Effective Action” by identifying h as the propagator of the heavy fields that are being integrated out.

我们可以通过将 h 识别为被积掉的重场的传播子，与“威尔逊有效作用量”一节的结果联系起来。

Homotopy Transfer for $\mathbf{L}_\infty$. This case has some new features, although the end result takes a similar form; here we will follow [73] with some changes of notation. Again we work with the tensor co-algebra $T(W)$ defined by the sum of symmetrized tensor products $SW^{\otimes n}$. We had before $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 + \cdots$. We will set $\mathbf{L} = \mathbf{b}$, write $\mathbf{Q} = \mathbf{b}_1$, and define $\ell = \mathbf{b}_2 + \mathbf{b}_3 + \cdots$. All in all, we have the degree one operator $\mathbf{L}$ given by

$\mathbf{L}_\infty$ 的同伦传递。该情况存在一些新特征，尽管最终结果形式相似；此处我们遵循文献 [73]，仅对记号做了一些改动。我们再次研究由对称张量积之和 $SW^{\otimes n}$ 定义的张量余代数 $T(W)$ 。我们之前已有 $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 + \cdots$ 。我们将设 $\mathbf{L} = \mathbf{b}$ ，写出 $\mathbf{Q} = \mathbf{b}_1$ ，并定义 $\ell = \mathbf{b}_2 + \mathbf{b}_3 + \cdots$ 。总而言之，我们得到一次算符 $\mathbf{L}$ 为

$$\mathbf{L} = \mathbf{Q} + \ell, \; \ell = \sum_{n=2}^{\infty} \mathbf{b}_n \tag{392}$$

The main identity satisfied by the products is $\mathbf{L}^2 = 0$, which implies

乘积满足的主要恒等式是 $\mathbf{L}^2 = 0$ ，由此可推出

$$\mathbf{Q}^2 = 0, \;\; \text{and} \;\; \mathbf{Q}\ell + \ell\mathbf{Q} + \ell^2 = 0. \tag{393}$$

We introduce the projector $P : W \to \bar{W}$, taking W to the subspace $\bar{W}$ of the effective field theory. We demand that

我们引入投影算符 $P : W \to \bar{W}$ ，它将 W 投影到有效场论的子空间 $\bar{W}$ 上。我们要求

$$P^2 = P,\ PQ = QP,\ (B_1, PB_2) = (PB_1, B_2)\,. \tag{394}$$

We extend this projector to a degree zero operator $\mathbf{P} : T(W) \to T(\bar{W})$. We write

我们将这个投影算符延拓为零次算符 $\mathbf{P} : T(W) \to T(\bar{W})$ ，写作

$$\mathbf{P}(B_1 \wedge B_2 \wedge \cdots \wedge B_n) \equiv PB_1 \wedge PB_2 \wedge \cdots \wedge PB_n. \tag{395}$$

so that $\mathbf{P}^2 = \mathbf{P}$. Moreover it follows from (394) that $\mathbf{P}$ and $\mathbf{Q}$ commute $\mathbf{QP} = \mathbf{PQ}$. As before, associated with the projector, there is a linear operator h of degree 1 satisfying [19]

使得 $\mathbf{P}^2 = \mathbf{P}$ 。此外由 (394) 可推得 $\mathbf{P}$ 与 $\mathbf{Q}$ 对易 $\mathbf{QP} = \mathbf{PQ}$ 。与之前一样，该投影算子对应一个次数为 1 的线性算子 h ，满足 [19]

$$hP = Ph = 0,\ h^2 = 0,\ Qh + hQ = 1 - P. \tag{396}$$

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[19] We have made a sign change on the right-hand side of the last equation, relative to the one used in [73]. This is in order to have L_∞ results analogous to the A_∞ results.

[19] 与文献 [73] 中的形式相比，我们对最后一个方程的右端改变了符号。这是为了得到 L_∞ 与 A_∞ 类似的结果。

The only new challenge here is defining $\mathbf{h}$, the associated odd operator on the symmetrized tensor algebra. This is done recursively. Using the notation $\bar{A} = PA$, we have

此处唯一的新问题是定义对称张量代数上的相关奇算子 $\mathbf{h}$ 。这一定义是递归完成的。采用记号 $\bar{A} = PA$ ，我们得到

$$h(A_1 \wedge \ldots \wedge A_n) \equiv \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma; A) \Big(h\left(A_{\sigma(1)} \wedge \ldots \wedge A_{\sigma(n-1)}\right) \wedge A_{\sigma(n)}$$
$$+ (-1)^{A_{\sigma(1)} + \ldots + A_{\sigma(n-1)}} \bar{A}_{\sigma(1)} \wedge \cdots \wedge \bar{A}_{\sigma(n-1)} \wedge hA_{\sigma(n)} \Big)\,.$$

(397)

The reader can check that this gives, for example,

读者可以验证，例如该定义给出

$$h(A_1 \wedge A_2) = \frac{1}{2}\Big(hA_1 \wedge A_2 + hA_1 \wedge \bar{A}_2 + (-1)^{A_1} \left(\bar{A}_1 \wedge hA_2 + A_1 \wedge hA_2\right)\Big). \tag{398}$$

The definition is such that the properties (396) holding on W get upgraded to identities in $T(W)$:

该定义的性质是，成立于 W 上的性质 (396) 可提升为 $T(W)$ 中的恒等式:

$$\mathbf{hP} = \mathbf{Ph} = 0,\ \mathbf{h}^2 = 0,\ \mathbf{Qh} + \mathbf{hQ} = \mathbf{1} - \mathbf{P}. \tag{399}$$

Since we have here the same identities on the tensor algebra as in the A_∞ case, the operator $\overline{\mathbf{L}}$ defining the new products is given by the analog of (387)

由于此处张量代数上的恒等式与 A_∞ 情形的完全相同，因此定义新乘积的算子 $\overline{\mathbf{L}}$ 由 (387) 的类比给出

$$\overline{\mathbf{L}} = \mathbf{PQP} + \mathbf{P}\ell \frac{1}{1 + \mathbf{h}\ell}\mathbf{P}, \tag{400}$$

and satisfies $\overline{\mathbf{L}}^2 = 0$, as required to be define L_∞ algebra. Moreover, $\overline{\mathbf{L}}$ is a coderivation in the tensor coalgebra, and the new string products are graded symmetric [73].

且满足定义 L_∞ 代数所需的 $\overline{\mathbf{L}}^2 = 0$ 。此外，$\overline{\mathbf{L}}$ 是张量余代数上的余导子，且新的弦乘积是分次对称的 [73]。

The analogs of (390) hold, in particular, $\overline{\ell}_3 = \mathbf{P}\left[\ell_3 - \ell_2\mathbf{h}\ell_2\right]\mathbf{P}$. Acting on string fields A_i , defining $\bar{A}_i = PA_i$, and $[A_1, A_2] = \ell_2(A_1 \wedge A_2)$, we have

(390) 的类比式成立，特别地，$\overline{\ell}_3 = \mathbf{P}\left[\ell_3 - \ell_2\mathbf{h}\ell_2\right]\mathbf{P}$ 。作用在弦场 A_i 上，定义 $\bar{A}_i = PA_i$ 和 $[A_1, A_2] = \ell_2(A_1 \wedge A_2)$ 后，我们得到

$$\bar{Q}A_1 = Q\bar{A}_1$$

$$\overline{\ell}_2(A_1 \wedge A_2) = P\left[\bar{A}_1, \bar{A}_2\right]$$

$$\overline{\ell}_3(A_1 \wedge A_2 \wedge A_3) = P\ell_3\left(\bar{A}_1 \wedge \bar{A}_2 \wedge \bar{A}_3\right)$$

$$-P\Big(\left[h\left[\bar{A}_1, \bar{A}_2\right], \bar{A}_3\right] + (-1)^{A_2A_3}\left[h\left[\bar{A}_1, \bar{A}_3\right], \bar{A}_2\right]$$

$$+\left[\bar{A}_1, h\left[\bar{A}_2, \bar{A}_3\right]\right]\Big)\ .$$

(401)

The expression for $\overline{\ell}_3$ is relevant to the computation of four-point functions. The last three terms are the contributions from three Feynman diagrams of the original field theory, where, due to h , only the states outside $\bar{W}$ propagate.

$\overline{\ell}_3$ 的表达式与四点函数的计算相关。最后三项是原场论中三个费曼图的贡献，其中由于 h ，仅有 $\bar{W}$ 外的态传播。

String Vertices

弦顶点

We have studied so far the algebraic structure of the various string field theories. These structures required a number of string products. These string products are defined, in general, using subsets of moduli spaces of Riemann surfaces with punctures and local coordinates at the punctures.

到目前为止，我们已经研究了各类弦场论的代数结构。这些结构需要若干弦乘积。一般而言，弦乘积是利用带孔黎曼面模空间的子集，以及孔处的局部坐标来定义的。

In this section, we will discuss in detail how these subsets of moduli spaces are selected and how we can introduce coordinates around the punctures. This is the information about string vertices. Consistency conditions on these choices are an equation quite analogous to the Batalin-Vilkovisky master equation. Equipped with such string vertices, we then discuss how such information allows us to write the string field theory interactions.

在本节中，我们将详细讨论如何选取这些模空间子集，以及如何在孔处引入坐标。这些就是关于弦顶点的信息。这些选取上的一致性条件是一个非常类似巴塔林-维尔可夫斯基主方程的方程。有了这样的弦顶点，我们接下来讨论这些信息如何帮助我们写出弦场论相互作用。

At the basis of all of this lies the concept of a BV algebra. It begins by defining a complex $\mathcal{C}$, usually an infinite dimensional vector space, whose elements form a graded-commutative, associative algebra under a simple multiplication called a dot product. Given two elements $X, Y \in \mathcal{C}$, we write the dot product as $X \cdot Y \equiv XY$ belonging to the same complex. Graded commutativity of the dot product means $XY = (-1)^{XY} YX$ where X, Y in the sign factor refer to the Grassmanality of X and Y . On the space $\mathcal{C}$, one introduces an odd operator Δ that squares to zero:

这一切的基础是 BV 代数的概念。它首先定义了一个复空间 $\mathcal{C}$, 通常是无限维向量空间, 其元素在称作点乘的简单乘法下构成阶化交换结合代数。给定两个元素 $X, Y \in \mathcal{C}$ ，我们将点乘记作 $X \cdot Y \equiv XY$ ，它属于同一个复空间。点乘的阶化交换性满足 $XY = (-1)^{XY} YX$ ，其中符号因子中的 X, Y 对应 X 和 Y 的格拉斯曼奇偶性。在空间 $\mathcal{C}$ 上，我们引入一个平方为零的奇算子 Δ :

$$\Delta (\Delta X) = \Delta^2 X = 0, \ \forall X \in \mathcal{C}. \tag{402}$$

Moreover, the operator Δ must be a second-order (super) derivation of the dot product:

此外，算子 Δ 必须是点乘的二阶 (超) 导子:

$$\Delta (XYZ) = \Delta (XY) Z + (-1)^X X \Delta (YZ) + (-1)^{XY+Y} Y \Delta (XZ)$$

(403)

$$-\Delta X (YZ) - (-1)^X X (\Delta Y) Z - (-1)^{X+Y} XY \Delta Z,$$

for X, Y , and Z arbitrary elements in $\mathcal{C}$ with definite Grassmanality. These properties define the BV algebra $(\mathcal{C}, \Delta)$. Still, one extra object, the antibracket $\{X, Y\}$ of two elements, can be defined from the above structure as the failure of Δ to be a first-order derivation:

其中 X, Y 和 Z 是 $\mathcal{C}$ 中具有确定格拉斯曼奇偶性的任意元素。这些性质定义了 BV 代数 $(\mathcal{C}, \Delta)$ 。从上述结构还可以定义额外的对象——两个元素的反括号 $\{X, Y\}$ ，它对应 Δ 不是一阶导子的偏离:

$$\{X, Y\} \equiv (-1)^{X} \Delta (XY) - (-1)^{X} (\Delta X)\, Y - X \Delta Y. \tag{404}$$

There are three properties of the antibracket that follow from the above definition: the exchange property, the Jacobi-like property, and the interaction with the dot product. These are quick to verify and read:

由上述定义可以推出反括号的三条性质: 交换性质、类雅可比性质，以及与点乘的相互作用性质。它们很容易验证，写作:

$$\begin{gathered}
\{X, Y\} = -(-1)^{(X+1)(Y+1)}\{Y, X\}, \\
0 = (-1)^{(X+1)(Z+1)}\{\{X, Y\}, Z\} + \text{ cyclic}, \\
\{X, YZ\} = \{X, Y\}Z + (-1)^{XY+Y} Y\{X, Z\}.
\end{gathered} \tag{405}$$

We will now explain how this BV structure is defined when $\mathcal{C}$ is built from Riemann surfaces with punctures, carrying a choice of local coordinates at each puncture. Then we will consider how this BV structure exists on functions defined on the vector space of a CFT. Finally, the relation between these structures will be established. This allows one to find an equation for string vertices that implies that the string field theory satisfies the BV master equation.

现在我们来解释，当 $\mathcal{C}$ 由带孔黎曼面构造，且每个孔都选定了局部坐标时，如何定义这个 BV 结构。随后我们会讨论该 BV 结构如何存在共形场论向量空间的函数上。最后我们会建立这些结构之间的关系。这让我们能够得到弦顶点满足的方程，由此可保证弦场论满足 BV 主方程。

For simplicity, we shall restrict most of our analysis to closed bosonic string field theory and will describe the results for open closed string field theory at the end. Generalization of this analysis to superstring field theory is straightforward.

为简单起见，我们将大部分分析限制在闭玻色弦场论，最后再描述开弦闭弦弦场论的结果。本文的分析推广到超弦场论是直接的。

BV Structures on Moduli Spaces

模空间上的 BV 结构

We first introduce the key definitions. We let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of genus g and n punctures. Punctures are labeled and are thus distinguishable. This moduli space is finite dimensional.

The space $\mathcal{P}_{g,n}$ is the moduli space of genus g surfaces and n punctures with a chosen analytic coordinate at each puncture. This space is infinite dimensional for $n > 0$, because one requires an infinite number of parameters to define an analytic coordinate around a puncture. The space $\mathcal{P}_{g,n}$ is a fibering over $\mathcal{M}_{g,n}$, with a projection that simply forgets about the analytic coordinates at the punctures. For closed string field theory, however, it is necessary to introduce a space $\widehat{\mathcal{P}}_{g,n}$, also fibered over $\mathcal{M}_{g,n}$. This space is obtained from $\mathcal{P}_{g,n}$ by a projection that forgets the phase of the local coordinate at each puncture.

我们首先引入核心定义。令 $\mathcal{M}_{g,n}$ 为亏格 g 带 n 个 puncture 的黎曼曲面模空间。Puncture 带有标记，因此是可区分的。该模空间是有限维的。空间 $\mathcal{P}_{g,n}$ 是亏格 g 带 n 个 puncture，且在每个 puncture 处选定了解析坐标的模空间。当 $n > 0$ 时该空间是无限维的，因为定义 puncture 附近的解析坐标需要无穷多个参数。空间 $\mathcal{P}_{g,n}$ 是 $\mathcal{M}_{g,n}$ 上的纤维丛，其投影直接忽略 puncture 处的解析坐标。但对于闭弦场论，我们还需要引入同样以 $\mathcal{M}_{g,n}$ 为底空间的纤维空间 $\widehat{\mathcal{P}}_{g,n}$ ，该空间由 $\mathcal{P}_{g,n}$ 通过投影抹去每个 puncture 处局部坐标的相位得到。

It is worth noting that local coordinate around a puncture is a map from a unit disk $|w| \leq 1$ to the surface, with $w = 0$ mapped to the puncture. The image of $|w| = 1$ under the map is a coordinate curve, a simple closed Jordan curve surrounding the puncture and homotopic to it. Specifying the phase of the local coordinate is equivalent to singling out a special point in the coordinate curve, say the image of $w = 1$. The coordinate curve with this special point marked is in fact enough, by the Riemann mapping theorem, to specify the local coordinate at the puncture. The coordinate curve, without any marked point, specifies the local coordinate up to a phase. This phase can be viewed as the marking of a special point on a closed string. We can therefore imagine any element of $\widehat{\mathcal{P}}_{g,n}$ as a Riemann surface of genus g , with n punctures and a coordinate curve around each puncture. A coordinate curve defines a coordinate disk-the disk that is the image of $|w| \leq 1$. Various coordinate disks should not have regions of overlaps. It is possible, however, for their boundaries to touch.

值得说明的是，puncture 处的局部坐标是从单位圆盘 $|w| \leq 1$ 到曲面的映射，其中 $w = 0$ 被映射到该 puncture。$|w| = 1$ 在映射下的像为坐标曲线，这是一条环绕 puncture 且与 puncture 同伦的简单闭若尔当曲线。确定局部坐标的相位等价于在坐标曲线上选定一个特殊点，例如 $w = 1$ 的像。根据黎曼映射定理，标记了特殊点的坐标曲线实际上足以确定 puncture 处的局部坐标；不带标记点的坐标曲线确定的局部坐标仅相差一个相位。这个相位可以看作是闭弦上一个特殊点的标记。因此我们可以将 $\widehat{\mathcal{P}}_{g,n}$ 的任意元素想象为一个亏格 g 带 n 个 puncture，且每个 puncture 附近都有一条坐标曲线的黎曼曲面。坐标曲线定义了坐标圆盘——即 $|w| \leq 1$ 的像。不同坐标圆盘不能有重叠区域，但它们的边界可以相切。

To build the complex $\mathcal{C}$, we consider finite dimensional subspaces $\mathcal{A}_{g,n}$ of $\widehat{\mathcal{P}}_{g,n}$. Note that these subspaces can have any dimension from zero to infinity when $n > 0$. Next we consider spaces of disconnected "generalized" surfaces as follows:

为了构造复形 $\mathcal{C}$ ，我们考虑 $\widehat{\mathcal{P}}_{g,n}$ 的有限维子空间 $\mathcal{A}_{g,n}$ 。注意当 $n > 0$ 时，这些子空间的维数可以是 0 到无穷大的任意值。接下来我们考虑不连通“广义”曲面的空间，构造如下:

$$\mathcal{A}_{g_1,n_1} \times \ldots \times \mathcal{A}_{g_r,n_r}. \tag{406}$$

A "surface" in this term is a disconnected one of the form $\left(\sum_{g_1,n_1}, \cdots \sum_{g_r,n_r}\right)$, in fact a collection of r surfaces, one from each subspace. Here, the disconnected surface has a total of $N = \sum_{i=1}^{r} n_i$ punctures. We assign a degree to the subspaces $\mathcal{A}_{g,n}$ by setting it equal to its dimensionality

这里所说的“曲面”是形如 $\left(\sum_{g_1,n_1},\cdots\sum_{g_r,n_r}\right)$ 的不连通曲面，实际上是一族 r 个曲面的集合，每个子空间各出一个曲面。该不连通曲面总共有 $N=\sum_{i=1}^{r} n_i$ 个 puncture。我们给子空间 $\mathcal{A}_{g,n}$ 赋予次数，次数等于其维数

$$\deg\left(\mathcal{A}_{g,n}\right)=\dim\left(\mathcal{A}_{g,n}\right)(\bmod 2). \tag{407}$$

The order of the factors in the above product matters only up to signs. Thus, for example,

上述乘积中因子的顺序仅影响符号。因此举个例子:

$$\mathcal{A}_{g_1,n_1}\times\mathcal{A}_{g_2,n_2}=(-1)^{\mathcal{A}_{g_1,n_1}\cdot\mathcal{A}_{g_2,n_2}}\mathcal{A}_{g_2,n_2}\times\mathcal{A}_{g_1,n_1}. \tag{408}$$

An important technical point has to do with labeling of punctures. In an element of $\mathcal{C}$, we need the punctures to be labeled from 1 to N . This is easily done by labeling the punctures successively from left to right: those in the first space from 1 to n_1 , those in the second space from n_1+1 to n_1+n_2 , and so on. Moreover, we need the element of $\mathcal{C}$ to be invariant under the exchange of labels-we can call this a symmetric space. This can be implemented above by applying to the surfaces in (406) a permutation operator $\mathbf{P}$ in the symmetric group S_N that permutes the N labels of the punctures and a normalization factor. The result is finally an element of the complex $\mathcal{C}$, denoted with double brackets [[...]]:

一个重要的技术问题与穿孔的标号有关。在 $\mathcal{C}$ 的一个元素中，我们需要给穿孔从 1 到 N 标号。这可以很容易地通过从左到右依次给穿孔标号完成: 第一个区间内的穿孔从 1 标到 n_1，第二个区间内的从 n_1+1 标到 n_1+n_2，依此类推。此外，我们要求 $\mathcal{C}$ 的元素在交换标号下保持不变——我们可以称其为对称空间。这可以通过对 (406) 中的曲面作用对称群 S_N 中的置换算符 $\mathbf{P}$ (该算符置换穿孔的 N 个标号) 再乘一个归一化因子来实现。最终结果是复形 $\mathcal{C}$ 中的一个元素，用双括号 [[...]] 表示如下:

$$[[\mathcal{A}_{g_1,n_1},\ldots,\mathcal{A}_{g_r,n_r}]]\equiv\frac{1}{n_1!\cdots n_r!}\sum_{\sigma\in S_N}\mathbf{P}_\sigma\left(\mathcal{A}_{g_1,n_1}\times\ldots\times\mathcal{A}_{g_r,n_r}\right)\in\mathcal{C}. \tag{409}$$

The normalization factor is such that if the $\mathcal{A}$ spaces are themselves symmetric, the only effect of the permutations is to move labels across the disconnected surfaces. The $[[\cdots]]$ are graded symmetric under the exchange of the $\mathcal{A}$ spaces within the brackets. The orientation of these $[[\cdots]]$ spaces is induced by the orientation of the $\mathcal{A}$ subspaces. The general element of $\mathcal{C}$ consists of linear superpositions of the above elements multiplied by real numbers.

若 $\mathcal{A}$ 空间本身是对称的，那么归一化因子会使得置换唯一的作用就是在不连通曲面之间移动标号。括号内的 $[[\cdots]]$ 在交换 $\mathcal{A}$ 空间下是分次对称的。这些 $[[\cdots]]$ 空间的定向由 $\mathcal{A}$ 子空间的定向诱导得到。$\mathcal{C}$ 的任意一般元素都是上述元素乘实数后得到的线性叠加。

Having defined $\mathcal{C}$, we must now define the dot product. This is actually easy; we take

定义好 $\mathcal{C}$ 后，我们现在必须定义点积。这其实很简单，我们取

$$\left[\left[\mathcal{A}_{g_1,n_1},\ldots,\mathcal{A}_{g_r,n_r}\right]\right]\cdot\left[\left[\mathcal{B}_{g_1',n_1'},\ldots,\mathcal{B}_{g_p',n_p'}\right]\right]$$

$$=\left[\left[\mathcal{A}_{g_1,n_1},\ldots,\mathcal{A}_{g_r,n_r},\mathcal{B}_{g_1',n_1'},\ldots,\mathcal{B}_{g_p',n_p'}\right]\right]. \tag{410}$$

This product is graded commutative, as expected, and is also manifestly associative. The unit element of this algebra is surfaces without any connected components and thus also without any punctures.

不出所料，这个乘积是分次交换的，也明显满足结合律。该代数的单位元是没有任何连通分量、因此也没有任何穿孔的曲面。

To complete the definition of the BV structure, we need to define the Δ operator. For this, we must make use of twist gluing introduced in (106). Consider two punctures P_i and P_j on a generalized surface; the punctures may lie on the same surface or on different surfaces. Let w_i and w_j denote the local coordinates around the punctures. Twist gluing of these two punctures means gluing the two punctures via the relation $w_i w_j = e^{i\theta}$, with $\theta \in [0, 2\pi]$. This operation reduces the number of punctures by two and increases the dimensionality by one, by adding the circle associated with the various values of θ . Since we consider the collection of all surfaces labelled by θ , this definition is insensitive to the choice of the phases of the local coordinates w_i and w_j and makes sense as operations on the elements of $\widehat{\mathcal{P}}_{g,n}$.

要完成 BV 结构的定义，我们需要定义 Δ 算符。为此，我们必须用到 (106) 中引入的扭转粘合。考虑广义曲面上的两个穿孔 P_i 和 P_j ；这两个穿孔可以位于同一个曲面上，也可以位于不同曲面上。设 w_i 和 w_j 分别是这两个穿孔附近的局部坐标。两个穿孔的扭转粘合指的是通过关系 $w_i w_j = e^{i\theta}$ 粘合这两个穿孔，满足 $\theta \in [0, 2\pi]$ 。该操作会将穿孔数量减少两个，并通过增加一个对应 θ 所有取值的圆将维数提高一。由于我们考虑的是由 θ 标号的所有曲面的集合，该定义对局部坐标 w_i 和 w_j 的相位选择不敏感，作为作用在 $\widehat{\mathcal{P}}_{g,n}$ 元素上的操作是良定义的。

We now define an operator Δ_{ij} with $i \neq j$ that acts on a product space $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_r$. We let $\Delta_{ij}\mathcal{A}$ equal $\frac{1}{2}$ times the set of surfaces obtained by twist gluing the punctures P_i and P_j for all the surfaces in $\mathcal{A}$. When the punctures lie on the same surface, the number of connected components of the generalized surface does not change, but the genus of the surface in question increases by one unit. When the punctures lie on two different surfaces, these are joined by the gluing, and the number of connected components decreases by one. In either case, the dimensionality of the space increases by one unit, with the twist angle parametrizing the new dimension. If the number of punctures in $\mathcal{A}$ is one or less, the action of Δ_{ij} gives zero. The orientation of $\Delta_{ij}A$ is defined by the ordered set of tangent vectors $\left[\frac{\partial}{\partial\theta}, \{\mathcal{A}\}\right]$, where $\{\mathcal{A}\}$ is the ordered collection of tangent vectors that define the orientation of $\mathcal{A}$.

我们现在定义一个作用在乘积空间 $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_r$ 上的算子 Δ_{ij} ，它满足 $i \neq j$ 。我们令 $\Delta_{ij}\mathcal{A}$ 等于 $\frac{1}{2}$ 乘以对 $\mathcal{A}$ 中所有曲面，将孔 P_i 与 P_j 扭转粘合后得到的曲面集合。当两个孔位于同一曲面上时，广义曲面的连通分支数不变，但该曲面的亏格增加 1。当两个孔位于两个不同曲面上时，粘合后两个曲面连接为一，连通分支数减少 1。两种情形下，空间的维数都增加 1，新增维度由扭转角参数化。若 $\mathcal{A}$ 中的孔数小于等于 1，则 Δ_{ij} 的作用结果为零。$\Delta_{ij}A$ 的定向由有序切向量集合 $\left[\frac{\partial}{\partial\theta}, \{\mathcal{A}\}\right]$ 定义，其中 $\{\mathcal{A}\}$ 是定义 $\mathcal{A}$ 定向的有序切向量集合。

More generally, we can now define the Δ operator as follows using the definition of the elements of $\mathcal{C}$. For any $X = \left[\left[\mathcal{A}_{n_1,g_1}, \cdots, \mathcal{A}_{g_r,n_r}\right]\right]$, we define [20]

更一般地，利用 $\mathcal{C}$ 中元素的定义，我们现在可以按如下方式定义 Δ 算子。对任意 $X = [[\mathcal{A}_{n_1,g_1},\cdots,\mathcal{A}_{g_r,n_r}]]$，我们定义 [20]

$$\Delta X \equiv \Delta_{ij} X \equiv \frac{1}{n_1!\cdots n_r!} \sum_{\sigma\in S_N} \Delta_{ij} \mathbf{P}_\sigma \left(\mathcal{A}_{n_1,g_1} \times \cdots \times \mathcal{A}_{g_r,n_r}\right), \tag{411}$$

with $i \neq j$ and $1 \leq i, j \leq N$. This definition is independent of the choice of i and j due to the symmetry of the space X .

满足 $i \neq j$ 和 $1 \leq i, j \leq N$ 。由于空间 X 的对称性，该定义与 i 和 j 的选取无关。

Let us now understand why the above-defined Δ operation satisfies $\Delta(\Delta X) = 0$. This is actually a matter of signs and can be understood by considering a generalized surface X with four or more punctures. Indeed, for three or less punctures, the first Δ reduces the number of punctures to one or less, and the second Δ simply gives zero. For four or more punctures, we will evaluate $\Delta^2 X$ in two ways, as $\Delta_{12}\Delta_{12}X$ and as $\Delta_{12}\Delta_{34}X$. The independence of Δ on the choice of punctures implies that the two evaluations should be the same. But we will show that they actually differ by a sign.

下面我们来理解上述定义的 Δ 运算为什么满足 $\Delta(\Delta X) = 0$ 。这实际上是一个符号问题，可以通过考虑一个带有四个或更多孔的广义曲面 X 来理解。事实上，对于三个及更少孔的情况，第一次 Δ 作用会将孔数减少到 1 或更少，因此第二次 Δ 作用结果直接为零。对于四个及更多孔的情况，我们将用两种方式计算 $\Delta^2 X$ ，分别得到 $\Delta_{12}\Delta_{12}X$ 和 $\Delta_{12}\Delta_{34}X$ 。Δ 对孔选取的独立性说明两种计算结果应该相等，但我们会证明它们实际上相差一个符号。

In calculating with the first option $\Delta_{12}\Delta_{12}X$, consider acting on an element $\sum_X \in X$. We first twist glue punctures P_1 and P_2 of $\sum_X$, then relabel the punctures $(P_3 \ldots P_N)$ as $(P_1 \ldots P_{N-2})$, and twist glue the new P_1 and P_2 punctures. Effectively, the second gluing operation is joining the original P_3 and P_4 punctures.

使用第一种方案 $\Delta_{12}\Delta_{12}X$ 计算时，考虑作用在元素 $\sum_X \in X$ 上。我们首先将 $\sum_X$ 的孔 P_1 和 P_2 扭转粘合，然后将孔 $(P_3 \ldots P_N)$ 重新标记为 $(P_1 \ldots P_{N-2})$ ，再对新的 P_1 和 P_2 孔做扭转粘合。实际上，第二次粘合操作连接的是原曲面的 P_3 和 P_4 孔。

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[20] The Δ operator here is what we called Δ_c in section "Closed Bosonic String Field Theory". Since at this stage we are dealing exclusively with closed strings, we drop the subscript c to avoid cluttering. Later, when we consider open closed string field theory, we shall have both Δ_c and Δ_o .

[20] 这里的 Δ 算子就是我们在“闭玻色弦场论”章节中所说的 Δ_c 。由于现阶段我们仅处理闭弦，因此我们去掉下标 c 以避免表述冗余。后续当我们讨论开弦闭弦弦场论时，我们会同时用到 Δ_c 和 Δ_o 。

Therefore, the orientation of the space $\Delta_{12}\Delta_{12}X$ at the subspace $\Delta_{12}\left(\Delta_{12}\sum_X\right)$ will contain the tangent vectors

因此，子空间 $\Delta_{12}\left(\Delta_{12}\sum_X\right)$ 上空间 $\Delta_{12}\Delta_{12}X$ 的定向将包含如下切向量

$$\Delta_{12}\left(\Delta_{12}\sum_X\right) \text{ orientation } \left[\frac{\partial}{\partial\theta_{34}}, \frac{\partial}{\partial\theta_{12}}, \{X\}\right], \tag{412}$$

where $\frac{\partial}{\partial\theta_{12}}$ is the tangent vector associated with the gluing of punctures one and two and $\frac{\partial}{\partial\theta_{34}}$ is the tangent vector associated with the gluing of punctures three and four. The ordering of the tangent vectors in the above expression is fixed as shown.

其中 $\frac{\partial}{\partial\theta_{12}}$ 是与孔 1 和孔 2 的粘合关联的切向量，$\frac{\partial}{\partial\theta_{34}}$ 是与孔 3 和孔 4 的粘合关联的切向量。上述表达式中切向量的顺序已按所示固定。

In calculating with the second option $\Delta_{12}\Delta_{34}X$, we again focus on the element $\sum_X \in X$. We first twist glue punctures P_3 and P_4 of $\sum_X$ and then twist glue the punctures P_1 and P_2 , which need no relabeling. This time, we have

在第二种选择 $\Delta_{12}\Delta_{34}X$ 下计算时，我们再次关注元素 $\sum_X \in X$ 。我们首先对 $\sum_X$ 的孔 P_3 和孔 P_4 进行扭粘合，随后对孔 P_1 和孔 P_2 进行扭粘合，该过程不需要重新标记。此时我们得到

$$\Delta_{12}\left(\Delta_{34}\sum_X\right) \text{ orientation } \left[\frac{\partial}{\partial\theta_{12}}, \frac{\partial}{\partial\theta_{34}}, \{X\}\right]. \tag{413}$$

We see that with this second option, we have the opposite orientation as in the first option. Given that the two ways of calculation Δ^2X give answers that differ by a minus sign, we conclude that $\Delta^2X = 0$.

我们可以看到，在第二种选择下，我们得到的定向与第一种选择相反。鉴于两种计算方式 Δ^2X 得到的结果相差一个负号，我们得出结论: $\Delta^2X = 0$ 。

The final property of Δ that must be shown is that it is a second-order derivation of the dot product, as indicated in (403). The verification of this relation is a matter of keeping track of the types of surfaces produced by the various terms. For example, we write $\Delta(XYZ) = R_{XX} + R_{YY} + R_{ZZ} + R_{XY} + R_{XZ} + R_{YZ}$, where R_{XX} , for example, denotes surfaces where the two punctures glued are in X and R_{XY} denotes surfaces where one puncture is in X and one is in Y . By tracking surfaces for each term on the right-hand side, one can prove that the equation holds. This requires some care with normalization factors and signs (see [108])

Δ 必须证明的最终性质是，它是点积的二阶导子，如式 (403) 所示。验证该关系只需跟踪各项产生的曲面类型即可。例如，我们记 $\Delta(XYZ) = R_{XX} + R_{YY} + R_{ZZ} + R_{XY} + R_{XZ} + R_{YZ}$ ，其中 R_{XX} 表示被粘合的两个孔都在 X 中的曲面，R_{XY} 表示被粘合的两个孔一个在 X 、一个在 Y 中的曲面。通过跟踪右侧每一项对应的曲面，即可证明该等式成立。这需要留意归一化因子和符号 (见 [108])

We now turn to the antibracket that as written in (404) is obtained as the failure of Δ to be a derivation of the dot product.

我们现在来讨论反括号，根据式 (404)，反括号由 Δ 不是点积导子的偏差得到。

$$\{X, Y\} \equiv (-1)^X \Delta(XY) - (-1)^X (\Delta X) Y - X\Delta Y. \tag{414}$$

For the first term on the right-hand side, we write

对于右侧的第一项，我们记

$$(-1)^X \Delta (XY) = S_{XX} + S_{YY} + S_{XY}, \tag{415}$$

where S_{XX} are the surfaces where two punctures in X are glued, S_{YY} are the surfaces where two punctures in Y are glued, and S_{XY} are the surfaces where the punctures glued are one in X and one in Y . All these surfaces have the orientation $(-1)^X\{\partial/\partial\theta, [X], [Y]\} = \{[X], \partial/\partial\theta, [Y]\}$. It also becomes clear that for the other two terms on the right-hand side,

其中 S_{XX} 是 X 中两个孔被粘合得到的曲面，S_{YY} 是 Y 中两个孔被粘合得到的曲面，S_{XY} 是被粘合的两个孔一个在 X 、一个在 Y 得到的曲面。所有这些曲面的定向都为 $(-1)^X\{\partial/\partial\theta, [X], [Y]\} = \{[X], \partial/\partial\theta, [Y]\}$ 。由此也可明确，对于右侧另外两项,

$$(-1)^X (\Delta X) Y = S_{XX}, \; X\Delta Y = S_{YY}, \tag{416}$$

with the sign factors manifestly coming from the orientations. We thus see that $\{X, Y\} = S_{XY}$ with orientation $\{[X], \partial/\partial\theta, [Y]\}$; the antibracket glues two punctures, one on the first entry and one on the second entry. In summary, if $\sum_X$ denotes an element of X and $\sum_Y$ denotes an element of Y , then $\{X, Y\}$ consists of surfaces where one puncture of $\sum_X$ is glued to one puncture of $\sum_Y$ and the final punctured surface is symmetrized on all the remaining punctures. It will often be the case that X and Y are just spaces with one connected component, and thus $\sum_X$ and $\sum_Y$ are just each a Riemann surface with punctures, as opposed to a generalized surface.

符号因子显然来自定向。因此我们得到 $\{X, Y\} = S_{XY}$ 的定向为 $\{[X], \partial/\partial\theta, [Y]\}$；反括号粘合两个孔，分别来自第一个条目和第二个条目。综上，若 $\sum_X$ 表示 X 的一个元素，$\sum_Y$ 表示 Y 的一个元素，则 $\{X, Y\}$ 由满足以下条件的曲面构成: $\sum_X$ 的一个孔与 $\sum_Y$ 的一个孔粘合，最终带孔曲面对所有剩余孔对称化。通常情况下，X 和 Y 都只是只有一个连通分支的空间，因此 $\sum_X$ 和 $\sum_Y$ 都各自只是一个带孔黎曼曲面，而非广义曲面。

Beyond the dot product and the Δ operator, we also have a boundary operator ∂ acting on $\mathcal{C}$. For any $X \in \mathcal{C}$, we write ∂X for the boundary of X . The boundary operator acting on a space of surfaces simply takes the boundary of the space, giving us another space of surfaces, with an orientation induced by the orientation of the original space. [21] Since ∂ decreases the dimensionality of a space by one unit, it changes degree. The operator ∂ is an odd derivation of the dot product

除了点积和 Δ 算符外，我们还有一个作用在 $\mathcal{C}$ 上的边界算符 ∂ 。对任意 $X \in \mathcal{C}$ ，我们将 X 的边界记为 ∂X 。作用在曲面空间上的边界算符仅取该空间的边界，得到另一个曲面空间，其定向由原空间的定向诱导得到。[21] 由于 ∂ 会将空间的维数降低 1，因此它会改变次数。算符 ∂ 是点积的一个奇导子

$$\partial (XY) = (\partial X) Y + (-1)^X X\partial Y. \tag{417}$$

Moreover ∂ anticommutes with Δ , as one can see by tracking the ordering of vectors defining the orientations

此外 ∂ 与 Δ 反对易，这一点可以通过追踪定义定向的矢量排序得到验证

$$\Delta\partial X = -\partial\Delta X. \tag{418}$$

Finally, using the definition of the antibracket from the Δ operator, one can quickly check that ∂ is an odd derivation of the antibracket:

最后，利用从 Δ 算符得到的反括号定义，可以快速验证 ∂ 是反括号的一个奇导子:

$$\partial\{X, Y\} = \{\partial X, Y\} + (-1)^{X+1}\{X, \partial Y\}. \tag{419}$$

The sign factor in the second term a consequence of the orientation $\{[X], \partial/\partial\theta, [Y]\}$ of $\{X, Y\}$, and to reach Y, ∂ goes "across" the X space and the gluing operation.

第二项中的符号因子是 $\{X, Y\}$ 的 $\{[X], \partial/\partial\theta, [Y]\}$ 定向带来的结果，为了到达 Y, ∂ ，需要“穿过” X 空间和粘合操作。

BV Structures on CFT

共形场论上的 BV 结构

The central point to be discussed here, focused on bosonic closed string field theory, is that for any matter CFT of $c = 26$ coupled to the reparameterization ghosts CFT, we can represent the BV structures discussed above on the space of functions of string fields. We call $\mathcal{H}$ the state space of this complete CFT.

本文聚焦玻色闭弦场论，核心讨论点为: 对任意耦合重参数化鬼场共形场论的 $c = 26$ 物质共形场论，我们可在弦场函数空间上表示前文讨论的 BV 结构。我们将 $\mathcal{H}$ 称为该完整共形场论的态空间。

As discussed in section "Bosonic String Amplitudes and Their Off-Shell Generalization", this conformal field theory supplies string field valued forms that can be integrated over moduli spaces of Riemann surfaces. More precisely, we have objects

正如“玻色弦振幅及其离壳推广”一节所述，该共形场论给出取值为弦场的形式，可在黎曼曲面模空间上积分。更准确地说，我们得到以下对象

$$\left\langle \Omega_k^{(g,n)} \right| \in (\mathcal{H}^*)^{\otimes n} \tag{420}$$

related to $\Omega_k^{(g,n)}(A_1, \cdots, A_n)$ introduced in (44) via

通过 (44) 式与引入的 $\Omega_k^{(g,n)}(A_1, \cdots, A_n)$ 相关联

$$\left\langle \Omega_k^{(g,n)} |A_1\rangle \otimes \cdots \otimes |A_n\right\rangle = \Omega_k^{(g,n)}(A_1, \cdots, A_n), \tag{421}$$

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[21] Given a point $p \in \mathcal{A}$, a set of basis vectors $[v_1, \ldots, v_k]$ in $T_p(\partial A)$ defines the orientation of ∂A if $[n, v_1, \ldots, v_k]$, with n an outward pointing basis vector of $T_p\mathcal{A}$, is the orientation of $\mathcal{A}$ at p .

[21] 给定一点 $p \in \mathcal{A}$ ，若当 n 是 $T_p\mathcal{A}$ 的外向基向量时，$[n, v_1, \ldots, v_k]$ 是 $\mathcal{A}$ 在 p 处的定向，则 $T_p(\partial A)$ 中的一组基向量 $[v_1, \ldots, v_k]$ 定义了 ∂A 的定向。

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that are differential forms of degree k in $\widehat{\mathcal{P}}_{g,n}$. These are suitable for integration over k dimensional subspaces $\mathcal{A}_{g,n}^{(k)}$ of $\widehat{\mathcal{P}}_{g,n}$. The forms above satisfy a very nontrivial property involving the BRST operator and the boundary operator on moduli spaces. We have

它们是 $\widehat{\mathcal{P}}_{g,n}$ 上次数为 k 的微分形式，适用于对 $\widehat{\mathcal{P}}_{g,n}$ 的 k 维子空间 $\mathcal{A}_{g,n}^{(k)}$ 积分。上述形式满足一个涉及 BRST 算子与模空间边界算子的非常不平凡的性质，我们有

$$\int_{\mathcal{A}_{g,n}} \left\langle \Omega_k^{(g,n} \middle| \sum_{i=1}^{n} Q^{(i)} = (-1)^k \int_{\partial\mathcal{A}_{g,n}} \left\langle \Omega_{k-1}^{(g,n)} \right| . \right\rangle \tag{422}$$

Suppose we have a symmetric space $\mathcal{A}_{g,n}^{(k)}$; we then define

给定对称空间 $\mathcal{A}_{g,n}^{(k)}$ ，我们定义

$$f\left(\mathcal{A}_{g,n}^{(k)}\right) \equiv \frac{1}{n!} \int_{\mathcal{A}_{g,n}^{(k)}} \left\langle \Omega_k^{(g,n)} |\Psi\rangle_1 \cdots |\Psi\rangle\right._n . \tag{423}$$

More generally, we have

更一般地，我们有

$$f\left(\left[\left[\mathcal{A}_{g_1,n_1}^{(k_1)} \cdots \mathcal{A}_{g_r,n_r}^{(k_r)}\right]\right]\right) \equiv \prod_{i=1}^{r} \frac{1}{n_i!} \int_{\mathcal{A}_{g_i,n_i}^{(k_i)}} \langle \Omega_{k_i}^{(g_i,n_i)} |\Psi\rangle_1 \cdots |\Psi\rangle_{n_i} . \tag{424}$$

This definition makes it clear that

该定义明确表明

$$f(XY) = f(X) f(Y), \; X, Y \in \mathcal{C}. \tag{425}$$

Operations Δ and $\{\cdot, \cdot\}$ on string functionals are defined as follows. We first define BPZ-implementing states $|R_{12}\rangle$ and $\langle R_{12}|$ via the relations

弦泛函上的运算 Δ 和 $\{\cdot, \cdot\}$ 定义如下: 我们首先通过以下关系定义实现 BPZ 的态 $|R_{12}\rangle$ 和 $\langle R_{12}|$

$$ {}_1\langle A\,|{}_2\langle B|\,R_{12}\rangle = \langle A\,|B\rangle,\ \langle R_{12}|\,A\rangle_1\,|B\rangle_2 = \langle A|\,B\rangle, \tag{426} $$

for any pair of states $|A\rangle,|\,B\rangle$ in the full CFT state space $\mathcal{H}'$. We than define symplectic forms $\langle\omega\ |$ and $|\ S\rangle$ as follows: [22]

对完整共形场论态空间 $\mathcal{H}'$ 中的任意一对态 $|A\rangle,|\,B\rangle$ 都成立。接着我们如下定义辛形式 $\langle\omega\ |$ 和 $|\ S\rangle$: [22]

$$ \langle\omega_{12}| = \langle R_{12}|\,c_0^{-(2)}, \tag{427} $$

$$ |S_{12}\rangle = b_0^{-(1)}\,|R_{12}\rangle\,. $$

---

[22] In [6, 80, 84] , the states $|R'_{12}\rangle$ and $\langle R'_{12}|$ defined as projections of $|R_{12}\rangle$ and $\langle R_{12}|$ to the kernel of L_0^- were used in (427). This is not needed when the symplecting forms act on states that are already projected, as is the case here.

[22] 在 [6, 80, 84] 中，作为 $|R_{12}\rangle$ 和 $\langle R_{12}|$ 向 L_0^- 核投影得到的态 $|R'_{12}\rangle$ 和 $\langle R'_{12}|$ 已用于 (427) 式。当辛形式作用于已经投影的态时 (正如本文中的情况)，这一步并不需要。

With these structures, we can now express the antibracket of two string functions F and G , defined in (116) and (117), as

有了这些结构，我们现在可以将 (116) 和 (117) 中定义的两个弦函数 F 和 G 的反括号表示为

$$ \{F,G\} = (-1)^{G+1}\frac{\partial F}{\partial|\Psi\rangle}\frac{\partial G}{\partial|\Psi\rangle}|S\rangle \tag{428} $$

Here the ket $|S\rangle$ glues the two state spaces left open by the differentiation with respect to the string field. They are $\langle F_R|\,c_0^-$ and $\langle G_R|\,c_0^-$ in the notation of (116). There is no need to specify left and right derivatives because the string field is even. The symplectic form can be used to write the kinetic term $S_{0,2}$ of the string field theory.

这里，右矢 $|S\rangle$ 粘合了对弦场求导后留下的两个态空间。沿用 (116) 的记号，它们是 $\langle F_R|\,c_0^-$ 和 $\langle G_R|\,c_0^-$ 。由于弦场是偶场，无需区分左右导数。利用辛形式可以写出弦场论的动力学项 $S_{0,2}$ 。

$$ S_{0,2} = \frac{1}{2}\langle\omega_{12}\,|Q^{(2)}|\,\Psi\rangle_1|\Psi\rangle_2. \tag{429} $$

For the Δ action on functions, we have

对于作用在函数上的 Δ ，我们有

$$\Delta F = \frac{1}{2}(-1)^{F+1}\left(\frac{\partial}{\partial|\Psi\rangle}\frac{\partial}{\partial|\Psi\rangle}F\right)|S\rangle. \tag{430}$$

We now have a wonderful interplay in which [108]

现在我们得到了美妙的相互作用，文献 [108] 指出

$$f(\Delta X) = -\Delta f(X), \tag{431}$$

where the Δ operation on the left-hand side acts on a space of surfaces and the Δ on the right-hand side is that acting on the space of string field functions. It now follows from the definition of the antibracket in terms of the Δ operators that

其中左侧的 Δ 作用于曲面空间，右侧的 Δ 作用于弦场函数空间。根据反括号基于 Δ 算子的定义，可得

$$f(\{X, Y\}) = -\{f(X), f(Y)\}, \tag{432}$$

where the antibracket on the right-hand side is that of string functionals.

其中右侧的反括号为弦泛函的反括号。

We also have that the boundary operator is implemented by the antibracket with the kinetic term of the string field theory:

我们还可知，边界算子由反括号与弦场理论动能项共同实现:

$$\left\{S_{0,2}, f(X)\right\} = -f(\partial X). \tag{433}$$

Geometric BV Master Equation and String Field Theory Master Equation

几何 BV 主方程与弦场论主方程

The string field theory vertices are, geometrically, collections of Riemann surfaces. The three-string vertex $\mathcal{V}_{0,3}$ of the classical theory is a Riemann sphere with three punctures and local coordinates, up to phases, at each of the punctures. The four-string vertex $\mathcal{V}_{0,4}$ consists of a subset of $\widehat{\mathcal{P}}_{0,4}$, namely, a collection of four punctured spheres, each with local coordinates, up to phases, at each of the punctures. In general, for the full quantum theory, we have vertices $\mathcal{V}_{g,n} \subset \widehat{\mathcal{P}}_{g,n}$, for various values of the genus g and the number of punctures n . We formally sum these sets of surfaces to form the full string vertex $\mathcal{V}$ defined as follows:

从几何角度看，弦场论顶点就是黎曼曲面的集合。经典理论中的三弦顶点 $\mathcal{V}_{0,3}$ 是一个带有三个孔的黎曼球面，且每个孔上都带有相差相位的局部坐标。四弦顶点 $\mathcal{V}_{0,4}$ 由 $\widehat{\mathcal{P}}_{0,4}$ 的一个子集构成，也就是一组带四个孔的球面，每个球面的每个孔上都带有相差相位的局部坐标。一般来说，对于完整量子理论，对亏格 g 和孔数 n 的各种取值，我们都有对应顶点 $\mathcal{V}_{g,n} \subset \widehat{\mathcal{P}}_{g,n}$。我们将这些曲面集合形式求和，得到完整弦顶点 $\mathcal{V}$，定义如下:

$$\mathcal{V} = \sum_{g,n} g_s^{2g+n-2} \mathcal{V}_{g,n}, \quad \text{with} \quad \begin{cases} n \geq 3, \ \text{for } g = 0, \\ n \geq 1, \ \text{for } g = 1, \\ n \geq 0, \ \text{for } g \geq 2. \end{cases} \tag{434}$$

The $g = 0$ vertices, belonging to the classical theory, begin with three punctures. The string vertex does not contain Riemann spheres with two or less punctures. The $g = 1$ vertices, associated with the leading quantum correction to the action, does not contain surfaces with zero punctures (the plain un-punctured torus). For $g \geq 2$, surfaces with all numbers of punctures, including zero, appear, although surfaces with no punctures give a constant contributions to the action. As such, they are not relevant in the verification that the action satisfies the master equation. These constant terms, however, are required for the string action to be background independent. Recalling that the Euler number of genus g surfaces with n punctures (viewed as boundaries) is $\chi_{g,n} = 2-2g-n$, we see that the moduli spaces enter $\mathcal{V}$ weighted by the factor $(g_s)^{-\chi_{g,n}}$, just as multilinear functions were built in (107). Interestingly, the list of vertices above is precisely that for which the surfaces have negative Euler number and thus admit hyperbolic metrics of constant negative curvature. The string vertices are symmetric under the exchange of punctures, a simple condition discussed in the context of our discussion of the BV algebra on moduli spaces.

属于经典理论的 $g = 0$ 顶点从三个孔开始。弦顶点不包含带有两个及更少孔的黎曼球面。与作用量领头量子修正相关的 $g = 1$ 顶点不包含零孔曲面 (即无孔环面)。对于 $g \geq 2$，包含零孔在内的各种孔数的曲面都会出现，尽管无孔曲面对作用量仅给出常数贡献。因此，它们在验证作用量满足主方程时并不相关。不过，这些常数项对于弦作用量满足背景独立性是必要的。回顾亏格 g 带有 n 个孔 (视为边界) 的曲面，其欧拉示性数为 $\chi_{g,n} = 2 - 2g - n$，我们可以看到模空间以权重因子 $(g_s)^{-\chi_{g,n}}$ 进入 $\mathcal{V}$，这和 (107) 中构造多线性函数的方式一致。有趣的是，上述顶点列表恰好对应欧拉示性数为负的曲面，这类曲面容许常负曲率双曲度量。弦顶点在交换孔的变换下不变，这是我们在讨论模空间上 BV 代数时提到的一个简单条件。

The $\mathcal{V}_{g,n}$ are often taken to be pieces of sections on the bundle $\widehat{\mathcal{P}}_{g,n}$ over $\mathcal{M}_{g,n}$. This condition, if satisfied, makes the construction more canonical, as each underlying Riemann surface in the vertex is constructed once and only once. Examples where this is achieved are constructions of vertices from minimal area metrics and the construction using hyperbolic metrics. It is clear, however, that the consistency of the string field theory is satisfied by vertices that are more general and are not strict sections. All that is needed is that $\mathcal{V}_{g,n}$ maps onto its image under the projection $\pi : \widehat{\mathcal{P}}_{g,n} \to \mathcal{M}_{g,n}$ with degree one. As explained in section "Bosonic String Amplitudes and Their Off-Shell Generalization" for the case of the full subspace $\mathcal{F}_{g,n}$, this means that a generic surface is counted once with multiplicity. On the left of Fig. 4, we show a string vertex $\mathcal{V}_{g,n}$ that is a piece of a section over $\mathcal{M}_{g,n}$. On the right we have a string vertex $\mathcal{V}_{g,n}$ that is not a piece of a section over $\mathcal{M}_{g,n}$. In both cases, the projection π maps $\mathcal{V}_{g,n}$ to its image with degree one. As discussed in section "Bosonic and Superstring Field Theories", $\mathcal{V}_{g,n}$ is part of $\mathcal{F}_{g,n}$; it consists of Riemann surfaces that are not produced by the Feynman diagrams with one or more propagators.

$\mathcal{V}_{g,n}$ 通常被视为 $\mathcal{M}_{g,n}$ 上从 $\widehat{\mathcal{P}}_{g,n}$ 的片段截面。如果满足这个条件，构造会更加典范，因为顶点中的每个基础黎曼曲面都被且仅被构造一次。满足该条件的例子包括极小面积度量构造顶点，以及双曲度量构造顶点。不过很明显，更一般的非严格截面顶点也能满足弦场论的自洽性。我们只需要 $\mathcal{V}_{g,n}$ 在投影 $\pi : \widehat{\mathcal{P}}_{g,n} \to \mathcal{M}_{g,n}$ 下以次数 1 映射到它的像即可。正如“玻色弦振幅及其离壳推广”一节中对完整子空间 $\mathcal{F}_{g,n}$ 的情况所解释的，这意味着一般曲面被计数一次，重数为 1。在图 4 的左侧，我们展示了弦顶点 $\mathcal{V}_{g,n}$ ，它是 $\mathcal{M}_{g,n}$ 上的一个片段截面；在右侧，我们展示了弦顶点 $\mathcal{V}_{g,n}$ ，它不是 $\mathcal{M}_{g,n}$ 上的片段截面。两种情况下，投影 π 都将 $\mathcal{V}_{g,n}$ 以次数 1 映射到其像。正如“玻色弦与超弦场论”一节中讨论的，$\mathcal{V}_{g,n}$ 是 $\mathcal{F}_{g,n}$ 的一部分；它由无法通过一个或多个传播子的费曼图生成的黎曼曲面构成。

String vertices, or just simply $\mathcal{V}$, must satisfy a consistency condition for the resulting string field theory to be consistent. This is a geometric version of the BV master equation, and it takes the form

弦顶点，或简称 $\mathcal{V}$ ，必须满足一个相容性条件，才能使由此得到的弦场理论自洽。这是 BV 主方程的几何版本，其形式为

$$\partial \mathcal{V} + \Delta \mathcal{V} + \frac{1}{2}\{\mathcal{V}, \mathcal{V}\} = 0, \tag{435}$$

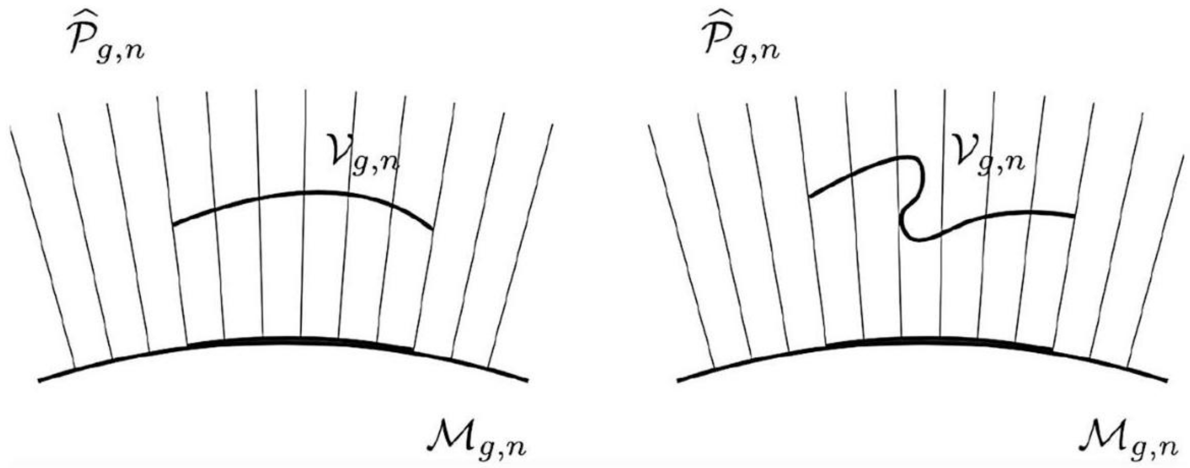


Fig. 4 A string vertex $\mathcal{V}_{g,n}$ that is a piece of a section over $\mathcal{M}_{g,n}$ (left) and a string vertex $\mathcal{V}_{g,n}$ that is not a piece of a section over $\mathcal{M}_{g,n}$ (right). In both cases, the projection π maps $\mathcal{V}_{g,n}$ to its image in $\mathcal{M}_{g,n}$ with degree one

图 4 一个是 $\mathcal{M}_{g,n}$ 上截面片段的弦顶点 $\mathcal{V}_{g,n}$ (左) 和一个不是 $\mathcal{M}_{g,n}$ 上截面片段的弦顶点 $\mathcal{V}_{g,n}$ (右)。两种情况下，投影 π 都将 $\mathcal{V}_{g,n}$ 以一次度映射到它在 $\mathcal{M}_{g,n}$ 中的像

We have discussed the objects in this equation before. We have an antibracket and a Δ operator in the space of surfaces and the boundary operator ∂ .

我们之前已经讨论过这个方程中的各个对象。在曲面空间中存在反括号与 Δ 算符，还有边界算符 ∂ 。

We have all the ingredients to show that the above geometric BV equation implies that the string field theory action S constructed using the vertex $\mathcal{V}$ satisfies the BV master equation in the space of string fields. The action is easily written down using the function f of string field constructed in the previous section using spaces of Riemann surfaces. We take

我们拥有全部要素可以证明，上述几何 BV 方程意味着，利用顶点 $\mathcal{V}$ 构造的弦场理论作用量 S 满足弦场空间中的 BV 主方程。利用前一节通过黎曼曲面空间构造的弦场函数 f ，可以很容易地写出该作用量。我们取

$$S = S_{0,2} + f(\mathcal{V}). \tag{436}$$

To verify the master equation, we first note that the kinetic term $S_{0,2}$ satisfies a couple of simple properties:

为验证主方程，我们首先注意到动能项 $S_{0,2}$ 满足若干简单性质:

$$\left\{S_{0,2}, S_{0,2}\right\} = 0,\ \Delta S_{0,2} = 0. \tag{437}$$

The first follows by direct computation (see [6]) and the second because the kinetic term couples vertex operators whose ghost numbers add up to four and those cannot be paired to a field and its associated antifield —in such pairs, the ghost numbers of the vertex operators add up to five. With these results, the verification of the master equation reduces to

第一条可通过直接计算得到 (参见文献 [6])，第二条是因为动能项耦合鬼数加和为四的顶点算符，而这类顶点算符无法配对为一个场和它对应的反场——在这类对中，顶点算符的鬼数加和为五。基于这些结果，主方程的验证可简化为

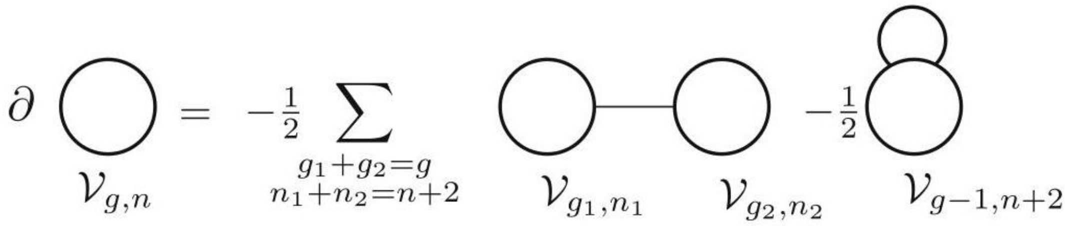
$$\begin{aligned}
0 &= \frac{1}{2}\{S, S\} + \Delta S \\
&= \left\{S_{0,2}, f(\mathcal{V})\right\} + \frac{1}{2}\{f(\mathcal{V}), f(\mathcal{V})\} + \Delta f(\mathcal{V}) \\
&= -f(\partial\mathcal{V}) - \frac{1}{2} f(\{\mathcal{V}, \mathcal{V}\}) - f(\Delta\mathcal{V}) \\
&= -f\left(\partial\mathcal{V} + \frac{1}{2}\{\mathcal{V}, \mathcal{V}\} + \Delta\mathcal{V}\right) = 0,
\end{aligned} \tag{438}$$

which is thus shown to hold on account of the geometrical BV equation satisfied by $\mathcal{V}$.

由此可证该式成立，因为 $\mathcal{V}$ 满足几何 BV 方程。

While having the string field action satisfy the BV master equation is the main reason for the geometrical BV equation, there are other aspects to this geometrical equation that are also quite important. In particular, focusing on the boundary $\partial\mathcal{V}_{g,n}$, we have the pictorial representation of the master equation

虽然让弦场作用量满足 BV 主方程是引入几何 BV 方程的主要原因，但这个几何方程的其他方面也十分重要。特别地，聚焦于边界 $\partial\mathcal{V}_{g,n}$ ，我们可以得到主方程的图示表示



The first term on the right-hand side shows the gluing pattern of the antibracket, joining two lower vertices whose surfaces, upon gluing, give surfaces of genus g and with n punctures, the same as the surface on the left-hand side. The second term shows the gluing pattern of the Δ operator acting on single surfaces. Since this operation increases the genus and reduces the number of punctures by two, the surfaces it acts upon belong to $\mathcal{V}_{g-1,n+2}$. The sets of surfaces on both sides of the equation must be the same as elements of $\widehat{\mathcal{P}}_{g,n}$. As an equation, the above reads

右侧第一项展示了反括号的粘合模式: 它连接两个更低阶顶点，粘合后得到亏格为 g 、带 n 个孔的曲面，和左侧的曲面相同。第二项展示了作用在单个曲面上的 Δ 算符的粘合模式。由于该操作会使亏格增加 1，孔数减少 2，因此它作用的曲面属于 $\mathcal{V}_{g-1,n+2}$ 。方程两侧的曲面集合作为 $\widehat{\mathcal{P}}_{g,n}$ 的元素必须完全相同。写成方程形式，上式为

$$\partial\mathcal{V}_{g,n} = -\Delta\mathcal{V}_{g-1,n+2} - \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n+2}} \{\mathcal{V}_{g_1,n_1}, \mathcal{V}_{g_2,n_2}\} \equiv \mathcal{O}_{g,n}. \tag{439}$$

Note that the twist gluing operation of local coordinates w_1, w_2 in $w_1w_2 = t$, with $|t| = 1$ is a boundary of the gluing operation $w_1w_2 = t$ with $|t| \le 1$. The latter creates a set of surfaces that is associated with a propagator joining the legs of the vertex. In this picture, the twist-gluing set represents the collapsed propagator, a propagator that has zero length. The $t = 0$ surface is a degenerate surface and corresponds to the limit of an infinitely long propagator. The relation displayed above tells us that the boundary of the vertex coincides with the collapsed propagator boundary of the Feynman diagrams built just with one propagator. We write this as

请注意，$w_1w_2 = t$ 中局部坐标的扭转粘合操作 w_1, w_2 (对应 $|t| = 1$) 是粘合操作 $w_1w_2 = t$ (对应 $|t| \le 1$) 的边界。后者得到的曲面集合对应连接顶点腿的传播子。在这个图像中，扭转粘合集合代表坍缩传播子，即长度为零的传播子。$t = 0$ 曲面是退化曲面，对应无穷长传播子的极限。上述关系表明，顶点的边界与仅用一个传播子构造的费曼图的坍缩传播子边界重合。我们将其写为

$$\partial\mathcal{V}_{g,n} = -\partial_p R_1, \tag{440}$$

where ∂_p denotes collapsed propagator boundary and R_1 is the set of surfaces built with lower vertices and just one propagator.

其中 ∂_p 表示坍缩传播子边界，R_1 是由更低阶顶点和仅一个传播子构造的曲面集合。

Deligne-Mumford Compactification, Noded Surfaces, and Feynman Diagrams

德利涅-芒福德紧化、节点曲面与费曼图

It is useful to pause to describe explicitly the degenerate surfaces and their role in the theory. Such surfaces, called noded surfaces, are the key element in the

我们不妨停下来明确描述退化曲面及其在理论中的作用。这类曲面被称为节点曲面，是

Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$. A noded surface is a connected complex space where points have neighborhoods complex isomorphic to $\{|z| < 1\}$, in which case they are regular points, or complex isomorphic to the set $\{zw = 0; |z| < 1, |w| < 1\}$ in $\mathbb{C}^2$, in which case they are nodes. We imagine a node as the single common point where two separate pieces of surfaces touch. Each component of the complement of the nodes, called a part of the noded surface, has negative Euler characteristic. A part can be a three-punctured sphere (counting the node as a puncture) but not, for example, a one or two punctured sphere. Depending on the number of nodes and their configuration, the surface can have one or more parts. If a surface has a single node, it may have one or two parts, depending on whether the degeneration is separating or nonseparating. It is also useful to describe the neighborhoods of the noded surfaces within the compactified space and so specify the topology of the compactification. For this, we consider the case of a noded surface $\sum_0$ with a single node and explain what it means to open up the node. Fix a complex number t with $|t| < \varepsilon$ small to remove a neighborhood $\mathcal{U}$ of the node $\mathcal{U} = \{|z| < |t|, |w| < |t|\}$, and form the identification space $\sum_t = \left(\sum_0 - \mathcal{U}\right)/(zw = t)$. This removes the node and connects locally the pieces of surfaces that before just touched via a plumbing fixture (note that the coordinates z and w must be defined all the way to $|z| = 1$ and $|w| = 1$). Surfaces near $\sum_0$ in the moduli space are (a) noded surfaces in which the parts are quasi-conformally close to the parts of $\sum_0$ and (b) smooth surfaces with small $t \neq 0$ obtained by opening up the node in the surfaces in (a). With this compactification, $\overline{\mathcal{M}}_{g,n}$ is now compact and boundaryless. Note that the string field theory construction of surfaces with propagators and plumbing fixtures $zw = t$ precisely describes the neighborhoods of the noded surfaces, with the limit $t \to 0$ as the noded surface. Noded surfaces are viewed as divisors in the moduli space.

模空间 $\mathcal{M}_{g,n}$ 的德利涅-芒福德紧化 $\overline{\mathcal{M}}_{g,n}$ 的核心要素。节点曲面是连通复空间，其点的邻域要么复同构于 $\{|z| < 1\}$，这类点是正则点；要么复同构于 $\mathbb{C}^2$ 中的集合 $\{zw = 0; |z| < 1, |w| < 1\}$，这类点是节点。我们可以将节点想象为两块曲面拼接处的公共单点。节点补集的每个分支称为节点曲面的一个分支，其欧拉特征为负。一个分支可以是三穿孔球面 (将节点计为一个穿孔)，但不能是单穿孔或双穿孔球面。根据节点数量与构型的不同，曲面可分为一个或多个分支。若曲面仅有一个节点，根据退化是分离型还是非分离型，它可分为一个或两个分支。描述紧化空间中节点曲面的邻域，从而明确紧化的拓扑结构也很有必要。为此，我们考虑仅含单个节点的节点曲面 $\sum_0$，并解释展开节点的含义。固定一个复数 t，其中 $|t| < \varepsilon$ 很小，移除节点 $\mathcal{U} = \{|z| < |t|, |w| < |t|\}$ 的邻域 $\mathcal{U}$，构造等同空间 $\sum_t = \left(\sum_0 - \mathcal{U}\right) / (zw = t)$。这一步会移除节点，并通过管道装置将原本仅接触的曲面分支局部连接起来 (注意坐标 z 和 w 必须在直至 $|z| = 1$ 和 $|w| = 1$ 处都有定义)。模空间中靠近 $\sum_0$ 的曲面分为两类:(a) 各分支拟共形逼近于 $\sum_0$ 各分支的节点曲面；(b) 对 (a) 中的节点展开节点得到的、带小 $t \neq 0$ 的光滑曲面。经过该紧化，$\overline{\mathcal{M}}_{g,n}$ 成为紧致无边界空间。注意弦场论中带传播子和管道装置的曲面构造 $zw = t$，恰好描述了节点曲面的邻域，极限 $t \to 0$ 就是节点曲面本身。节点曲面可视为模空间中的除子。

The vertices in $\mathcal{V}$ must be sufficient to construct, with Feynman rules, all the Riemann surfaces in the various moduli spaces-this is, after all, the way string amplitudes must be generated. In general, after using string vertices together with a propagator to form Feynman graphs of string field theory, the result is some submanifold $\mathcal{F}_{g,n}$ in $\widehat{\mathcal{P}}_{g,n}$. This is clear, because gluing of surfaces in $\widehat{\mathcal{P}}_{g,n}$ gives surfaces in $\widehat{\mathcal{P}}_{g,n}$. It is traditionally required that $\mathcal{F}_{g,n}$ should be a full section of the bundle $\widehat{\mathcal{P}}_{g,n}$, so that the map $\mathcal{F}_{g,n} \to \mathcal{M}_{g,n}$ is a homeomorphism. As will be discussed in the following sections, this can actually be achieved with minimal area string vertices and almost surely with hyperbolic string vertices. It is not necessary, however. All that is required is that the map $\mathcal{F}_{g,n} \to \mathcal{M}_{g,n}$ be of degree one. Such a map is surjective. This is what colloquially would be referred as "covering the moduli space." This is enough to imply that the integral over $\mathcal{F}_{g,n}$ of any differential form pulled back from $\mathcal{M}_{g,n}$ will be the same as the integral over $\mathcal{M}_{g,n}$. Since the integrand for on-shell string states is a top form on $\mathcal{M}_{g,n}$, this implies that the amplitude for on-shell string states computed by integrating over $\mathcal{M}_{g,n}$ coincides with that computed using string field theory Feynman rules. More precisely, and for clarity, we work with the compactified moduli space $\overline{\mathcal{M}}_{g,n}$ and we also view $\mathcal{F}_{g,n}$ as a space without boundary. However, as will be discussed shortly and in more detail in section "Applications of String Field Theory", we shall encounter several situations where the degenerate surfaces require special treatment.

$\mathcal{V}$ 中的顶点必须足以通过费曼规则构造出所有模空间中的所有黎曼曲面——毕竟这是生成弦振幅的必经方式。一般来说，将弦顶点与传播子结合构造弦场论的费曼图后，得到的结果是 $\widehat{\mathcal{P}}_{g,n}$ 中的某个子流形 $\mathcal{F}_{g,n}$。这一点很明确，因为 $\widehat{\mathcal{P}}_{g,n}$ 中的曲面粘合后得到的仍是 $\widehat{\mathcal{P}}_{g,n}$ 中的曲面。传统上要求 $\mathcal{F}_{g,n}$ 是丛 $\widehat{\mathcal{P}}_{g,n}$ 的一个完全截面，因此映射 $\mathcal{F}_{g,n} \to \mathcal{M}_{g,n}$ 是同胚。我们将在后续章节讨论，这一点实际上可以用极小面积弦顶点实现，几乎可以肯定双曲弦顶点也能实现。但这一要求并不是必要的，我们只需要求映射 $\mathcal{F}_{g,n} \to \mathcal{M}_{g,n}$ 是一度映射即可。这样的映射是满射，也就是通俗所说的“覆盖模空间”。这足以保证，从 $\mathcal{M}_{g,n}$ 拉回的任意微分形式在 $\mathcal{F}_{g,n}$ 上的积分都等于其在 $\mathcal{M}_{g,n}$ 上的积分。由于 on-shell 弦态的被积函数是 $\mathcal{M}_{g,n}$ 上的最高次形式，这说明在 $\mathcal{M}_{g,n}$ 上积分得到的 on-shell 弦态振幅，与用弦场论费曼规则计算得到的振幅一致。为了清晰起见，我们更精确地说，我们讨论的是紧化模空间 $\overline{\mathcal{M}}_{g,n}$，并且我们也将 $\mathcal{F}_{g,n}$ 视为无边界空间。不过我们很快就会讨论，且在“弦场论的应用”一节中会更详细说明，我们会遇到多种需要特殊处理退化曲面的情况。

In string field theory, we build $\mathcal{F}_{g,n}$ as follows:

在弦场论中，我们按如下方式构造 $\mathcal{F}_{g,n}$：

$$\mathcal{F}_{g,n} = \mathcal{V}_{g,n} \oplus R_1 \oplus \cdots R_{3g-3+n}. \tag{441}$$

Here R_I is a the set of surfaces built using I propagators. As indicated above, the maximum number of propagators is $3g-3+n$. Since $\mathcal{F}_{g,n}$ is boundaryless, a closed chain in mathematical language, the space to the right must be boundaryless. There are two types of boundaries that appear in this construction: (i) the boundaries of the vertices and (ii) the boundaries from surfaces where the length of a propagator is zero. When the length of the propagators is infinite, we approach the divisors, noded surfaces that are not counted as boundaries. The two kinds of boundaries cancel each other by the geometric master equation. In fact, one can show that the vertex boundaries of R_I cancel against the propagator boundary of R_{I+1} , this being an extension of (440). Note also that R_{3g-3+n} has no vertex boundary, as it is a region built by gluing a set of $\mathcal{V}_{0,3}$ that have no moduli.

此处 R_I 是使用 I 个传播子构造出的曲面集合。如上所述，传播子的最大数量为 $3g-3+n$ 。由于 $\mathcal{F}_{g,n}$ 无边界，用数学语言来说是一个闭链，因此右侧的空间必须无边界。该构造会产生两类边界:(i) 顶点的边界，(ii) 传播子长度为零的曲面带来的边界。当传播子长度为无穷大时，我们趋近于除子，即节点曲面，这类不算作边界。通过几何主方程，这两类边界会相互抵消。事实上，可以证明 R_I 的顶点边界会与 R_{I+1} 的传播子边界抵消，这是 (440) 的推广。还需要注意，R_{3g-3+n} 不存在顶点边界，因为它是由一组无模 $\mathcal{V}_{0,3}$ 粘合得到的区域。

Infinite-length propagators leading to noded surfaces require special attention. If the momentum propagating along the propagator is generic and can be varied continuously, then by choosing the momentum appropriately, we can make the forms $\Omega_k^{(g,n)}$ fall off sufficiently rapidly near these regions. The amplitudes for other values of momenta for which $\Omega_k^{(g,n)}$ does not fall off near these regions can be defined by analytic continuation. A systematic procedure of carrying out this analytic continuation was described in [39]. There are however cases where the momentum along the propagator is not generic, e.g., when it is forced to be either zero or on-shell due to momentum conservation. In such cases, the contribution from these regions requires special attention. This will be the subject of our discussion in section ”Applications of String Field Theory”. In this section, we shall proceed by assuming that $\Omega_k^{(g,n)}$ falls off sufficiently fast near these regions.

产生节点曲面的无限长传播子需要特殊处理。如果沿传播子传播的动量是通有的且可以连续变化，那么通过适当选择动量，我们可以使形式 $\Omega_k^{(g,n)}$ 在这些区域附近足够快地衰减。对于 $\Omega_k^{(g,n)}$ 在这些区域附近不衰减的其他动量取值，其振幅可以通过解析延拓定义。文献 [39] 中描述了执行这种解析延拓的系统步骤。但也存在沿传播子的动量不是通有的情况，例如，由于动量守恒，它被迫取零或在壳取值。在这类情况下，这些区域的贡献需要特殊处理。这将是我们在“弦场论的应用”一节中讨论的主题。在本节中，我们将假设 $\Omega_k^{(g,n)}$ 在这些区域附近足够快地衰减，在此基础上继续讨论。

It is useful to consider string vertices $\mathcal{V}_{g,n}$ that are not submanifolds $\widehat{\mathcal{P}}_{g,n}$ but rather more general singular chains of degree $6g-6+2n$. By using singular chains, the space $\mathcal{V}_{g,n}$ could be the formal sum of several disjoint spaces or could have self-intersections, for example. Since the purpose of introducing string vertices is to integrate over them and there is a well-behaved theory of integration of differential forms on singular chains, these kinds of vertices are allowed. As we will see, due to complications with picture changing operators in superstring field theory, string vertices are often singular chains. In such cases, $\mathcal{F}_{g,n}$ is also a singular chain. Because of the symmetry under permutation of the punctures, it is an S_n invariant chain.

考虑弦顶点 $\mathcal{V}_{g,n}$ 不是子流形 $\widehat{\mathcal{P}}_{g,n}$ 而是次数为 $6g-6+2n$ 的更一般奇异链是很有用的。例如，使用奇异链时，空间 $\mathcal{V}_{g,n}$ 可以是若干不相交空间的形式和，也可以存在自交。由于引入弦顶点的目的是对其积分，而微分形式在奇异链上有性质良好的积分理论，因此允许这类顶点。正如我们将要看到的，由于超弦场论中图变算子的复杂性，弦顶点通常都是奇异链。在这类情况下，$\mathcal{F}_{g,n}$ 也是奇异链。由于打孔置换对称性，它是一个 S_n 不变链。

The precise statement of covering of moduli space that leads to correct on-shell string amplitudes is that the chain $\mathcal{F}_{g,n}$, constructed as Feynman diagrams using $\mathcal{V}$ and a propagator, when projected via the forgetting of the coordinates, must represent the fundamental homology class of $\overline{\mathcal{M}}_{g,n}$. As discussed in [34], this constraint will be satisfied if the chain $\mathcal{V}$ satisfies the geometric master equation (435).

能得到正确的在壳弦振幅的模空间覆盖的精确表述是: 利用 $\mathcal{V}$ 和传播子按费曼图构造得到的链 $\mathcal{F}_{g,n}$ ，在忘记坐标投影后，必须表示 $\overline{\mathcal{M}}_{g,n}$ 的基本同调类。正如文献 [34] 中讨论的，如果链 $\mathcal{V}$ 满足几何主方程 (435)，则该约束成立。

In solving the geometrical master equations to find $\mathcal{V}$, the starting point is the vertex $\mathcal{V}_{0,3}$, the three-punctured sphere. Since all three punctured spheres are conformally equivalent, we just need an assignment of local coordinates at the punctures respecting the symmetry under exchange of punctures. The simplest cases from the geometrical master equation arise from its constraints on vertices of complex dimension one: these are $\mathcal{V}_{0,4}$ and $\mathcal{V}_{1,1}$. We have

求解几何主方程寻找 $\mathcal{V}$ 时，出发点是顶点 $\mathcal{V}_{0,3}$ 即三穿孔球面。由于所有三穿孔球面都是共形等价的，我们只需要在保持打孔交换对称性的前提下对穿孔的局部坐标赋值。几何主方程最简单的情况来自它对复维数 1 顶点的约束: 这些顶点就是 $\mathcal{V}_{0,4}$ 和 $\mathcal{V}_{1,1}$ 。我们有

$$\partial\mathcal{V}_{0,4} = -\frac{1}{2}\{\mathcal{V}_{0,3}, \mathcal{V}_{0,3}\}, \text{ and } \partial\mathcal{V}_{1,1} = -\Delta\mathcal{V}_{0,3}. \tag{442}$$

The next step is to find the sets $\mathcal{V}_{0,4}$ and $\mathcal{V}_{1,1}$ that satisfy the above conditions. As we will discuss concretely, this is always possible. One includes in the sets all the surfaces that lie in the interior of the moduli spaces up to the prescribed boundaries. On these surfaces, the local coordinates must be chosen as known on the boundaries and continuously within the set.

下一步是寻找满足上述条件的集合 $\mathcal{V}_{0,4}$ 和 $\mathcal{V}_{1,1}$ 。我们将具体说明，这总是可行的。我们将所有位于模空间内部直到规定边界的曲面都包含在集合中。在这些曲面上，局部坐标必须满足边界上已知，且在集合内部连续选取。

Existence of String Vertices

弦顶点的存在性

While we need explicit solutions of the geometric BV equations in order to do off-shell computations in string field theory, a more basic question is whether there exists a solution to these equations. This was answered in the affirmative by Costello [79], with the full details of this argument given in [34]. We briefly sketch the main points of the inductive argument.

我们需要几何 BV 方程的显式解才能开展弦场论的离壳计算，但一个更基础的问题是这些方程是否存在解。Costello[79] 给出了肯定的答案，该论证的完整细节记载于文献 [34]。我们简要概括这个归纳论证的要点。

Suppose we have found all the vertices $\mathcal{V}_{g',n'}$ such that $3g'-3+n' < 3g-3+n$. At this point, we need to find $\mathcal{V}_{g,n}$ satisfying

假设我们已经找到了所有满足 $3g'-3+n' < 3g-3+n$ 的顶点 $\mathcal{V}_{g',n'}$ 。此时我们需要找到满足下式的 $\mathcal{V}_{g,n}$:

$$\partial\mathcal{V}_{g,n} = \mathcal{O}_{g,n}. \tag{443}$$

The right-hand side $\mathcal{O}_{g,n}$ of this equation was defined in (439), and it is $(6g- \ 6+2n)-1$ chain on $\widehat{\mathcal{P}}_{g,n}$. It is built entirely from $\mathcal{V}_{g',n'}$, which we have already constructed. By construction, $\mathcal{O}_{g,n}$ is an S_n invariant chain, so we require $\mathcal{V}_{g,n}$ to be S_n invariant as well. It is a simple calculation to show that $\partial\mathcal{O}_{g,n}=0$; it only requires using the master equation for the spaces we have already constructed. This is a consistency condition for the above equation: the chain $\mathcal{O}_{g,n}$ must be closed. But this does not suffice, the chain should be trivial so that we can solve for $\mathcal{V}_{g,n}$. In other words, the appropriate homology class of $\mathcal{O}_{g,n}$ must vanish:

该方程的右端项 $\mathcal{O}_{g,n}$ 已在式 (439) 中定义，它是定义在 $\widehat{\mathcal{P}}_{g,n}$ 上的 $(6g- \ 6+2n)-1$ 链，完全由我们已经构造好的 $\mathcal{V}_{g',n'}$ 构建而成。根据构造，$\mathcal{O}_{g,n}$ 是一条 S_n 不变链，因此我们要求 $\mathcal{V}_{g,n}$ 也满足 S_n 不变性。简单计算即可得到 $\partial\mathcal{O}_{g,n}=0$ ，只需要用到我们已经构造好的空间的主方程。这就是上述方程的相容性条件: 链 $\mathcal{O}_{g,n}$ 必须是闭链。但这还不够，该链必须是零调的，我们才能解出 $\mathcal{V}_{g,n}$ 。换句话说，$\mathcal{O}_{g,n}$ 对应的同调类必须为零:

$$\left[\mathcal{O}_{g,n}\right] \in H_{6g-6+2n-1}\Big(\widehat{\mathcal{P}}_{g,n}\Big)^{S_n} \text{ must vanish.} \tag{444}$$

The first step in proving this is to show that $\widehat{\mathcal{P}}_{g,n}$ is homotopy equivalent to $\mathcal{M}_{g,n}$. This implies that the homology groups of $\widehat{\mathcal{P}}_{g,n}$ and of $\mathcal{M}_{g,n}$ are isomorphic. This step is somewhat technical. One first shows that $\widehat{\mathcal{P}}_{g,n}$ is isomorphic to a space $\widehat{\mathcal{P}}'_{g,n}$ of Riemann surfaces of genus g with n boundary components and an angular coordinate, defined up to rotation, on each boundary. This latter space is actually known to be homotopy equivalent of $\mathcal{M}_{g,n}$, establishing the result.

证明该结论的第一步是证明 $\widehat{\mathcal{P}}_{g,n}$ 同伦等价于 $\mathcal{M}_{g,n}$ 。这说明 $\widehat{\mathcal{P}}_{g,n}$ 和 $\mathcal{M}_{g,n}$ 的同调群同构。这一步略显技术化。首先可以证明 $\widehat{\mathcal{P}}_{g,n}$ 同构于空间 $\widehat{\mathcal{P}}'_{g,n}$ ，该空间由亏格为 g 、带有 n 个边界分量、每个边界上定义了一个旋转等价的角坐标的黎曼曲面组成。已知该空间同伦等价于 $\mathcal{M}_{g,n}$ ，由此得证。

The second step consists in showing that $H_{6g-6+2n-1}\left(\left(\mathcal{M}_{g,n}\right)^{S_n}\right)$ is zero. In fact, a stronger result is true: the homology $H_{6g-6+2n-1}\left(\mathcal{M}_{g,n}\right)$ vanishes for all spaces except for $(g,n)=(0,4)$. This is shown by identifying $H_{6g-6+2n-1}\left(\mathcal{M}_{g,n}\right)$, via Poincare duality, with the first cohomology class $H^1\left(\overline{\mathcal{M}}_{g,n}, \overline{\mathcal{M}}_{g,n}/\mathcal{M}_{g,n}\right)$ of the compactified moduli spaces, relative to the nodal surfaces in the compactification. An exact sequence of relative cohomology groups leads to the desired result. This means that the homology $H_{6g-6+2n-1}\left(\left(\mathcal{M}_{g,n}\right)^{S_n}\right)$ vanishes as well for these spaces.

第二步是证明 $H_{6g-6+2n-1}\left(\left(\mathcal{M}_{g,n}\right)^{S_n}\right)$ 等于零。事实上，可以得到一个更强的结论: 除 $(g,n)=(0,4)$ 外，所有空间的同调 $H_{6g-6+2n-1}\left(\mathcal{M}_{g,n}\right)$ 都为零。这一结论通过庞加莱对偶将 $H_{6g-6+2n-1}\left(\mathcal{M}_{g,n}\right)$ 等同于紧化模空间相对于紧化中结点曲面的上同调类 $H^1\left(\overline{\mathcal{M}}_{g,n}, \overline{\mathcal{M}}_{g,n}/\mathcal{M}_{g,n}\right)$ 来证明。相对上同调群的正合序列给出了我们想要的结果。这说明对于这些空间，同调 $H_{6g-6+2n-1}\left(\left(\mathcal{M}_{g,n}\right)^{S_n}\right)$ 也为零。

In the case of the four-punctured sphere $(g,n)=(0,4)$, the unrestricted one-dimensional homology group $H_1\left(\mathcal{M}_{0,4}\right)$ does not vanish, but the S_4 -invariant subgroup does. This is intuitively clear: the first homology group of $\mathcal{M}_{0,4}$ is two-dimensional, with representative cycles where the last marked point z_4 moves around 0 (the location of z_1) or moves around 1 (the location of z_2). This is clear: for each of these cycles, there is no region of the sphere without a puncture for which the cycle is a boundary. There is no independent cycle where z_4 moves around the point at ∞ (the location of z_3). On the other hand, a symmetrized cycle would have to be symmetric also under the S_3 subgroup of S_4 that permutes the first three punctures. This cycle would require three closed curves: one around 0, one around 1, and one around ∞ . They can be chosen not to intersect. It is clear then that there is a region for which this cycle is a boundary, showing that the symmetric homology group vanishes.

在四点球面 $(g,n)=(0,4)$ 的情况下，无限制一维同调群 $H_1\left(\mathcal{M}_{0,4}\right)$ 并不为零，但 S_4 不变子群为零。这一点直观上很清晰: $\mathcal{M}_{0,4}$ 的一阶同调群是二维的，其代表闭链对应最后一个标记点 z_4 绕 0(z_1 的位置) 运动，或绕 1(z_2 的位置) 运动。不难发现: 对这些闭链中的任意一个，球面都不存在不带穿孔的区域能使得该闭链成为其边界，也不存在 z_4 绕 ∞ 处点 (z_3 的位置) 的独立闭链。另一方面，对称化闭链还需要对置换前三个穿孔的 S_4 的 S_3 子群保持对称。这个对称化闭链需要三条闭曲线: 一条绕 0，一条绕 1，一条绕 ∞ ，且可选择让它们互不相交。不难看出，此时存在区域使得该闭链成为其边界，说明对称同调群为零。

Similar results for open-closed bosonic string theory have been discussed in [109] and those for heterotic string theory and superstring theory have been discussed in [110].

开-闭玻色弦理论的类似结论已在文献 [109] 中讨论, 杂弦理论和超弦理论的相关结论已在文献 [110] 中讨论。

Uniqueness of String Vertices up to Canonical Transformations

弦顶点在正则变换下的唯一性

Solutions of the master equation $\Delta S+\frac{1}{2}\{S,S\}=0$ are not unique. The infinitesimal canonical transformation

主方程 $\Delta S+\frac{1}{2}\{S,S\}=0$ 的解不唯一。无穷小正则变换

$$\delta S=\Delta\varepsilon+\{S,\varepsilon\} \tag{445}$$

leaves the master equation unchanged to order ε . This new action is equivalent to the original one. We now show that solutions of the geometric BV master equation are also not unique in a similar way. There is a notion of a canonical transformation of the vertex $\mathcal{V}$ that generates a new vertex that also satisfies the geometric master equation. This is relatively simple to verify for infinitesimal canonical transformations, as we do now.

使主方程在 ε 阶保持不变。新作用量与原作用量等价。下面我们将说明，几何 BV 主方程的解也存在类似的非唯一性。顶点 $\mathcal{V}$ 的正则变换可以生成同样满足几何主方程的新顶点。我们接下来就会看到，对于无穷小正则变换，这一点验证起来相对简单。

If we have some collection of $(6g-6+2n+1)$ -chains $\mathcal{W}_{g,n}$ of $\widehat{\mathcal{P}}_{g,n}$, then we can vary the string vertices $\mathcal{V}_{g,n}$ by

如果我们拥有 $(6g-6+2n+1)$ 链 $\mathcal{W}_{g,n}$ 构成的集合，这些链属于 $\widehat{\mathcal{P}}_{g,n}$ ，那么我们可以按如下方式改变弦顶点 $\mathcal{V}_{g,n}$:

$$\delta_{\mathcal{W}}\mathcal{V}=\{\mathcal{V},\mathcal{W}\}+\Delta\mathcal{W}+\partial\mathcal{W}, \tag{446}$$

where, as before, $\mathcal{V}=\sum g_s^{2g+n-2}\mathcal{V}_{g,n}, \mathcal{W}=\sum g_s^{2g-n+2}\mathcal{W}_{g,n}$. We claim that the new vertices, obtained by the canonical transformation,

和之前一样，此处 $\mathcal{V}=\sum g_s^{2g+n-2}\mathcal{V}_{g,n}, \mathcal{W}=\sum g_s^{2g-n+2}\mathcal{W}_{g,n}$ 。我们认为，通过正则变换得到的新顶点

$$\mathcal{V}+\delta\mathcal{W}\mathcal{V} \tag{447}$$

satisfy the quantum master equation, to leading order in $\delta_{\mathcal{W}}\mathcal{V}$: [23]

在 $\delta_{\mathcal{W}}\mathcal{V}$: [23] 领头阶满足量子主方程

$$\frac{1}{2}\{\mathcal{V}+\delta_{\mathcal{W}}\mathcal{V},\mathcal{V}+\delta_{\mathcal{W}}\mathcal{V}\}+\Delta(\mathcal{V}+\delta_{\mathcal{W}}\mathcal{V})+\partial(\mathcal{V}+\delta_{\mathcal{W}}\mathcal{V}) \tag{448}$$

$$=\left\{\partial\mathcal{V}+\Delta\mathcal{V}+\frac{1}{2}\{\mathcal{V},\mathcal{V}\},\mathcal{W}\right\}+O\left((\delta_{\mathcal{W}}\mathcal{V})^2\right)=0.$$

Two sets of string vertices related by geometric canonical transformations like this yield string field actions related by canonical transformation. Indeed, recalling that $S_{\mathcal{V}}=S_{0,2}+f(\mathcal{V})$, it now follows from that

通过此类几何正则变换联系起来的两组弦顶点，给出的弦场作用量也由正则变换联系。实际上，回忆 $S_{\mathcal{V}}=S_{0,2}+f(\mathcal{V})$ ，由此立刻可以得到

$$S_{\mathcal{V}+\delta_{\mathcal{W}}\mathcal{V}}=S_{\mathcal{V}}+f(\delta_{\mathcal{W}}\mathcal{V})=S_{\mathcal{V}}+f(\{\mathcal{V},\mathcal{W}\}+\Delta\mathcal{W}+\partial\mathcal{W})$$

$$=S_{\mathcal{V}}-\{f(\mathcal{V}),f(\mathcal{W})\}-\Delta f(\mathcal{W})-\left\{S_{0,2},\mathcal{W}\right\} \tag{449}$$

$$= S_{\mathcal{V}} - \{S_{\mathcal{V}}, f(\mathcal{W})\} - \Delta f(\mathcal{W}).$$

We see that the variation of the action fits the structure shown in (445), with $\varepsilon = -f(\mathcal{W})$, showing that the action for the new vertices is obtained by an infinitesimal canonical transformation of the original action.

我们可以看到，作用量的变分符合 (445) 给出的结构，其中 $\varepsilon = -f(\mathcal{W})$，这说明新顶点对应的作用量正是原作用量经过无穷小正则变换得到的。

The converse of the above results are important. We showed that an infinitesimal canonical transformation of $\mathcal{V}$ yields a vertex that also satisfies the master equation. It is possible to show that given any two solutions of $\mathcal{V}$ and $\mathcal{V}'$ of the BV master equation, one can construct a large canonical transformation that relates the two [34]. The setup for the proof is as follows.

上述结论的逆命题十分重要。我们已经证明，$\mathcal{V}$ 的无穷小正则变换会得到同样满足主方程的顶点。可以证明，给定 BV 主方程的任意两个解 $\mathcal{V}$ 和 $\mathcal{V}'$，我们都可以构造一个大正则变换将二者联系起来 [34]。证明的框架如下。

As a first step, we define "large" canonical transformations as some kind of exponential of small ones. Let $\mathcal{V}$ be a set of string vertices and $\mathcal{W}$ a collection $\mathcal{W}_{g,n}$ of $(6g-6+2n+1)$-dimensional S_n-invariant singular chains in $\widehat{\mathcal{P}}_{g,n}$, which define an infinitesimal canonical transformation. We define a family $\mathcal{V}(t)$ of string vertices by requiring $\mathcal{V}(0) = \mathcal{V}$ and demanding that they satisfy the differential equation

第一步，我们将“大”正则变换定义为小正则变换的某种指数形式。设 $\mathcal{V}$ 是一组弦顶点，$\mathcal{W}$ 是 $\mathcal{W}_{g,n}$ 的集合，由 $\widehat{\mathcal{P}}_{g,n}$ 中 $(6g-6+2n+1)$ 维 S_n 不变奇异链组成，这些奇异链定义了一个无穷小正则变换。我们通过条件 $\mathcal{V}(0) = \mathcal{V}$ 要求弦顶点满足微分方程，从而定义弦顶点的一个族 $\mathcal{V}(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(t) = \delta\mathcal{W}\mathcal{V}(t) \tag{450}$$

where $\delta_{\mathcal{W}}\mathcal{V}$ was defined above. If $\mathcal{V}$ satisfies the master equation, so does $\mathcal{V}(t)$ for all t. To show this, we write the master equation for $\mathcal{V}(t)$ in the form $M_{\mathcal{V}}(t) = 0$ by defining

其中 $\delta_{\mathcal{W}}\mathcal{V}$ 已在上文定义。若 $\mathcal{V}$ 满足主方程，则对任意 t，$\mathcal{V}(t)$ 也满足主方程。为证明这一点，我们通过如下定义将 $\mathcal{V}(t)$ 的主方程写为 $M_{\mathcal{V}}(t) = 0$ 的形式:

$$M_{\mathcal{V}}(t) \equiv \partial\mathcal{V}(t) + \Delta\mathcal{V}(t) + \frac{1}{2}\{\mathcal{V}(t), \mathcal{V}(t)\}. \tag{451}$$

Clearly $M_{\mathcal{V}}(0) = 0$ since $\mathcal{V}$ satisfies the master equation. A short calculation using the differential equation (450) shows that

显然由于 $\mathcal{V}$ 满足主方程，因此有 $M_{\mathcal{V}}(0) = 0$。利用微分方程 (450) 做简短计算即可得到

$$\frac{\mathrm{d}M_{\mathcal{V}}}{\mathrm{d}t} = \{M_{\mathcal{V}}(t), \mathcal{W}\} \tag{452}$$

[23] Useful identities $\Delta\{X,Y\}=\{\Delta X,Y\}+(-1)^{X+1}\{X,\Delta Y\}$ and $(-1)^{(X_1+1)(X_3+1)}\{\{X_1,X_2\}, X_3\}+\text{cyclic}=0$. All ∂,Δ , and $\{\cdot,\cdot\}$ change degree by one unit.

> [23] 有用恒等式 $\Delta\{X,Y\}=\{\Delta X,Y\}+(-1)^{X+1}\{X,\Delta Y\}$ 和 $(-1)^{(X_1+1)(X_3+1)}\{\{X_1,X_2\}, X_3\}+$ 循环 $=0$ 。所有 ∂,Δ ，且 $\{\cdot,\cdot\}$ 的次数改变 1 个单位。

This equation implies that if $M_{\mathcal{V}}(0)=0$, all derivatives of $M_{\mathcal{V}}(t)$ will vanish at $t=0$. This shows $M_{\mathcal{V}}(t)$ vanishes at all t . We write the solution of (450) for the instantaneous vertices by taking multiple derivatives, evaluating at $t=0$, and writing the Taylor series. One finds

> 该方程表明若 $M_{\mathcal{V}}(0)=0$ ，则 $M_{\mathcal{V}}(t)$ 的所有导数在 $t=0$ 处均为零。这说明 $M_{\mathcal{V}}(t)$ 在所有 t 处都等于零。我们通过求多重导数、在 $t=0$ 处取值并写出泰勒级数，得到了瞬时顶点的方程 (450) 的解。可得

$$\mathcal{V}(t)=\mathcal{V}+t\delta_{\mathcal{W}}\mathcal{V}+\frac{1}{2}t^2\{\delta_{\mathcal{W}}\mathcal{V},\mathcal{W}\}+\frac{1}{3!}t^3\{\{\delta_{\mathcal{W}}\mathcal{V},\mathcal{W}\},\mathcal{W}\}+\cdots. \tag{453}$$

We define the exponential of the canonical transformation via $\exp(t\delta_{\mathcal{W}})\mathcal{V}\equiv\mathcal{V}(t)$. Note that the series solution does not fit the naive expansion of the exponential; we do not encounter nor define iterated variations $\delta_{\mathcal{W}}\delta_{\mathcal{W}}\mathcal{V}$. The definition implies that

> 我们通过 $\exp(t\delta_{\mathcal{W}})\mathcal{V}\equiv\mathcal{V}(t)$ 定义正则变换的指数。注意级数解不符合指数的朴素展开；我们既不遇到也不定义迭代变分 $\delta_{\mathcal{W}}\delta_{\mathcal{W}}\mathcal{V}$ 。该定义表明

$$\exp(\delta_{\mathcal{W}})\mathcal{V}\equiv\mathcal{V}+\delta_{\mathcal{W}}\mathcal{V}+\frac{1}{2}\{\delta_{\mathcal{W}}\mathcal{V},\mathcal{W}\}+\frac{1}{3!}\{\{\delta_{\mathcal{W}}\mathcal{V},\mathcal{W}\},\mathcal{W}\}+\cdots. \tag{454}$$

Having defined large canonical transformations, the uniqueness of vertices is argued as follows. Suppose that $\mathcal{V}_{g,n}$ and $\mathcal{V}'_{g,n}$ are two sets of string vertices, both of which satisfy the master equation. The goal is to show that there exists a sequence of S_n -invariant singular chains $\mathcal{W}_{g,n}$ such that their exponentiation—a large canonical transformation-relates the two sets of vertices

> 定义了大正则变换后，我们按如下方式论证顶点的唯一性。假设 $\mathcal{V}_{g,n}$ 和 $\mathcal{V}'_{g,n}$ 是两组弦顶点，二者都满足主方程。我们的目标是证明存在一系列 S_n 不变奇异链 $\mathcal{W}_{g,n}$ ，其指数化——即一个大正则变换——可以将两组顶点联系起来

$$\exp(\delta_{\mathcal{W}})\mathcal{V}=\mathcal{V}'. \tag{455}$$

The construction of $\mathcal{W}$ is done by induction (for details, see [34], Section 2.3). The initial step deals with the lowest genus zero vertex, the three punctured sphere. Here $\mathcal{V}_{0,3}$ and $\mathcal{V}'_{0,3}$ are both S_3 invariant points in $\widehat{\mathcal{P}}_{0,3}$. Since this space is connected, we let $\mathcal{W}_{0,3}$ be a S_3 -invariant path connecting $\mathcal{V}_{0,3}$ to $\mathcal{V}'_{0,3}$. Viewing $\mathcal{W}_{0,3}$ as a one-chain, we have $\partial\mathcal{W}_{0,3}=\mathcal{V}'_{0,3}-\mathcal{V}_{0,3}$. This implies that we have satisfied (455) to leading order. The full induction argument is a bit intricate and uses the vanishing of the homology groups $H_{6g-6+2n}\left(\widehat{\mathcal{P}}_{g,n}\right)$ for all (g,n) except $(0,3)$.

$\mathcal{W}$ 的构造通过归纳法完成 (详见文献 [34] 第 2.3 节)。归纳的初始步骤处理最低亏格的零顶点，即三孔球面。此处 $\mathcal{V}_{0,3}$ 和 $\mathcal{V}'_{0,3}$ 都是 $\widehat{\mathcal{P}}_{0,3}$ 中 S_3 的不变点。由于该空间是连通的，我们令 $\mathcal{W}_{0,3}$ 为连接 $\mathcal{V}_{0,3}$ 和 $\mathcal{V}'_{0,3}$ 的 S_3 不变路径。将 $\mathcal{W}_{0,3}$ 视为一维链，可得 $\partial\mathcal{W}_{0,3} = \mathcal{V}'_{0,3} - \mathcal{V}_{0,3}$ 。这说明我们在领头阶满足了式 (455)。完整的归纳论证稍显复杂，它用到除 $(0,3)$ 外对所有 (g,n) 同调群 $H_{6g-6+2n}\left(\widehat{\mathcal{P}}_{g,n}\right)$ 都是零这一性质。

Open-Closed Vertices and Their Main Identity

开弦-闭弦顶点及其主恒等式

The main geometric identity for open-closed string field theory was given in (156). The pictorial representation is given in Fig. 5, where, on the first line, we show the three ways in which two punctures on the same surface can be glued together via the complete Δ operator. This includes Δ_c for closed strings, Δ'_o for open strings on the same boundary component, and Δ_o for open string on different boundary components. The closed string antibracket and the open string antibracket are on the second and third lines, respectively.

开弦-闭弦弦场论的主几何恒等式最早出现在文献 (156) 中。其图示表示见图 5，第一行展示了同一曲面上两个孔通过完整的 Δ 算子粘在一起的三种方式: 分别对应闭弦的 Δ_c 、同一边界分量上开弦的 Δ'_o ，以及不同边界分量上开弦的 Δ_o 。闭弦反括号与开弦反括号分别在第二行和第三行。

The various open-closed vertices are combined to form the open closed chain $\mathcal{V}$ as follows:

各类开弦-闭弦顶点组合得到开弦闭弦链 $\mathcal{V}$ ，形式如下:

$$\mathcal{V} = \sum_{g,b,n_c,n_o} (g_s)^{-\chi_{g,b,n_c,n_o}} \mathcal{V}_{g,b,n_c,n_o}, \tag{456}$$

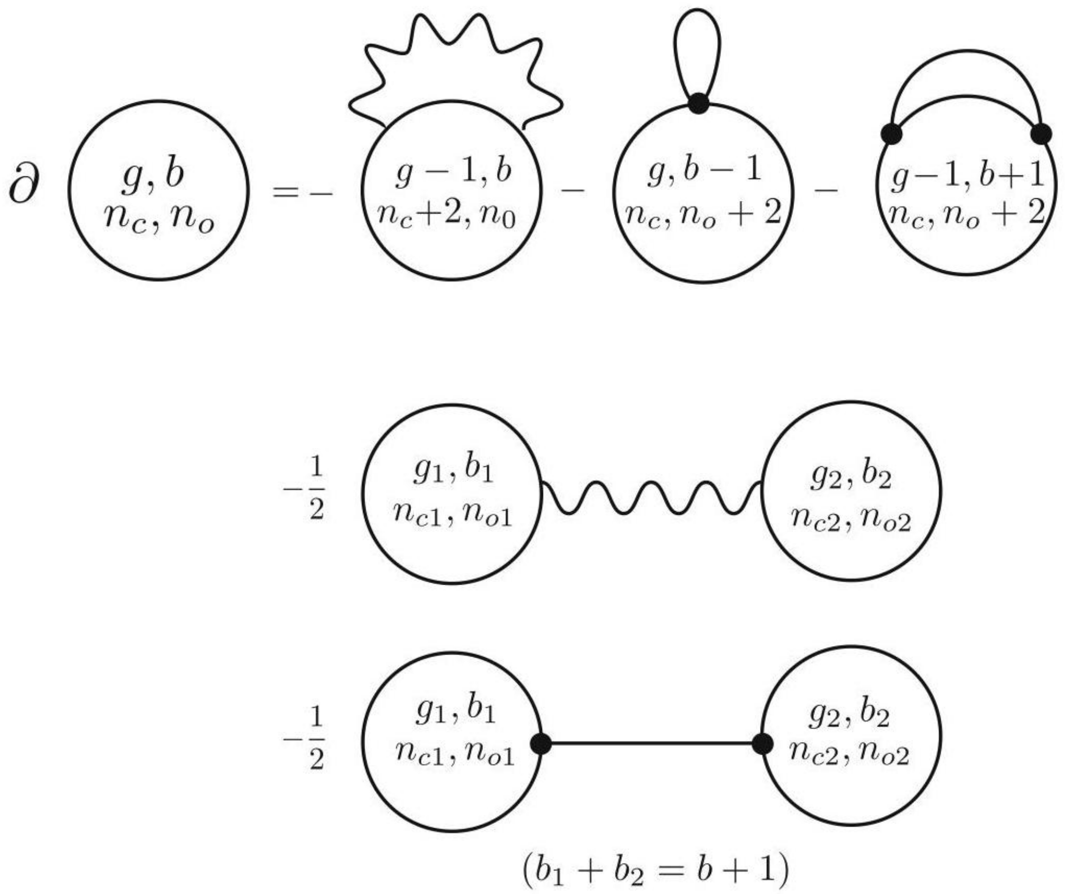


Fig. 5 The boundary of a general open-closed vertex coincides with a closed string Δ operator acting on a vertex, two types of open string Δ operators acting on vertices, and closed and open antibrackets of vertices. Wavy lines represent gluing of closed string punctures; continuous lines represent gluing of open string punctures. Heavy dots represent open string boundaries, and external open and closed states are not shown

图 5 一般开弦-闭弦顶点的边界，等于作用在顶点上的一个闭弦 Δ 算子、作用在顶点上的两类开弦 Δ 算子，以及顶点的闭弦反括号和开弦反括号之和。波浪线代表闭弦孔的粘合；实线代表开弦孔的粘合。粗点代表开弦边界，未画出外部开弦态与闭弦态

with $\chi_{g,b,n_c,n_o} = 2-2g-n_c-b-\frac{1}{2}n_o$, the Euler number of the surfaces in the associated vertex. Surfaces appearing in the above sum are those for which the Euler number is negative, with the exception of the disk with one closed string puncture and the annulus without punctures, both of which have Euler number zero. [24] Not included in $\mathcal{V}$ are therefore spheres ($g = b = 0$) with $n_c \leq 2$ and the torus ($g = 1, b = 0$) with $n_c = 0$, both of which are not included in the pure closed string, and the disks ($g = 0, b = 1, n_c = 0$) with $n_o \leq 2$. The above open-closed chain satisfies the familiar geometric master equation

其中 $\chi_{g,b,n_c,n_o} = 2-2g-n_c-b-\frac{1}{2}n_o$ 为对应顶点中曲面的欧拉示性数。上述求和中包含的曲面都是欧拉示性数为负的曲面，例外情况是带一个闭弦孔的圆盘和不带孔的环面，二者欧拉示性数均为零。[24] 因此 $\mathcal{V}$ 中不包含带 $n_c \le 2$ 的球面 $(g=b=0)$ 和带 $n_c=0$ 的环面 $(g=1,b=0)$ ——这二者都不包含在纯闭弦理论中，也不包含带 $n_o \le 2$ 的圆盘 $(g=0,b=1,n_c=0)$ 。上述开弦闭弦链满足我们熟知的几何主方程

$$\partial\mathcal{V} + \Delta\mathcal{V} + \frac{1}{2}\{\mathcal{V},\mathcal{V}\} = 0, \tag{457}$$

which is formally identical to the one considered for closed string theory. Here, of course, $\Delta = \Delta_c + \Delta_o' + \Delta_o$, as discussed above, and for the antibracket, we have both closed and open sector contributions: $\{\cdot,\cdot\} = \{\cdot,\cdot\}_c + \{\cdot,\cdot\}_o$.

它形式上和闭弦理论中讨论的主方程完全一致。当然此处正如前文所述，这里的 $\Delta = \Delta_c + \Delta_o' + \Delta_o$ ，并且对于反括号，我们同时有闭弦扇区和开弦扇区的贡献: $\{\cdot,\cdot\} = \{\cdot,\cdot\}_c + \{\cdot,\cdot\}_o$ 。

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[24] As in the case of purely closed string theory, surfaces without punctures give constant contribution to the string field theory action and do not affect any physical quantity computed from the theory.

[24] 和纯闭弦理论的情况一样，不带孔的曲面对弦场论作用量只贡献常数，不会改变从该理论计算得到的任何物理量。

In this open-closed main identity, we can identify subsets of vertices that satisfy consistent sub-identities under the possible operations [9]. The first two below are ”obvious,” the others less so.

在这个开弦-闭弦主恒等式中，我们可以找出在给定操作下满足自洽子恒等式的顶点子集 [9]。前两种是“显然的”，其余的则不那么明显。

1. Pure closed string theory. Setting $b = n_o = 0$ in (456), we get a chain $\mathcal{V}_c$ containing all closed string vertices that satisfy

1. 纯闭弦理论。在 (456) 中令 $b = n_o = 0$ ，我们得到包含所有满足条件的闭弦顶点的链 $\mathcal{V}_c$

$$\partial\mathcal{V}_c + \Delta\mathcal{V}_c + \frac{1}{2}\{\mathcal{V}_c,\mathcal{V}_c\}_c = 0. \tag{458}$$

Note that no antibracket or delta operation on surfaces with boundaries can produce a surface without boundaries. The chain of classical closed string vertices $\widehat{\mathcal{V}}_c = \sum_{n=3}^{\infty} (g_s)^{n-2}\mathcal{V}_{0,n,0,0}$ satisfies

注意，对带边界曲面的反括号或 delta 操作都不可能生成不带边界的曲面。经典闭弦顶点链 $\widehat{\mathcal{V}}_c = \sum_{n=3}^{\infty} (g_s)^{n-2}\mathcal{V}_{0,n,0,0}$ 满足

$$\partial\widehat{\mathcal{V}}_c + \frac{1}{2}\left\{\widehat{\mathcal{V}}_c, \widehat{\mathcal{V}}_c\right\}_c = 0. \tag{459}$$

2. Pure classical open string theory. Setting $g = 0, n_c = 0$ and $b = 1$ in (456), we get a chain $\mathcal{V}_o$ that includes disks with three or more open string punctures. Note that the Δ operator in the open string sector cannot produce surfaces in this class: Δ_o' increases the number of boundaries, and we cannot begin with $b = 0$; Δ_o increases genus, so the genus cannot remain zero. It follows that we have

2. 纯经典开弦理论。在 (456) 中令 $g = 0, n_c = 0$ 和 $b = 1$ ，我们得到链 $\mathcal{V}_o$ ，它包含带有三个及以上开弦孔的圆盘。注意，开弦扇区的 Δ 算子无法生成这类曲面: Δ_o' 会增加边界数量，而我们一开始没有 $b = 0$ ； Δ_o 会增加亏格，因此亏格无法保持为零。由此我们得到

$$\partial\mathcal{V}_o + \frac{1}{2}\{\mathcal{V}_o, \mathcal{V}_o\}_o = 0. \tag{460}$$

3. $\mathcal{V}_1$ consisting of disks with $n_c = 1$ and $n_o \geq 0$. Here $\mathcal{V}_1 = \sum_{n_o=0}^{\infty} (g_s)^{\frac{1}{2}n_o} \mathcal{V}_{0,1,1,n_o}$. Inspection of Fig. 5 shows that the boundary of this chain interacts with the classical open string chain $\mathcal{V}_o$ as follows:

3. $\mathcal{V}_1$ 由带有 $n_c = 1$ 和 $n_o \geq 0$ 的圆盘构成。此处为 $\mathcal{V}_1 = \sum_{n_o=0}^{\infty} (g_s)^{\frac{1}{2}n_o} \mathcal{V}_{0,1,1,n_o}$ 。观察图 5 可知，该链的边界与经典开弦链 $\mathcal{V}_o$ 的相互作用如下:

$$\partial\mathcal{V}_1 + \{\mathcal{V}_o, \mathcal{V}_1\}_o = 0. \tag{461}$$

4. $\mathcal{V}_2$ consisting of disks with $n_c = 2$ and $n_o \geq 0$. Here $\mathcal{V}_2 = \sum_{n_o=0}^{\infty} (g_s)^{1+\frac{1}{2}n_o} \mathcal{V}_{0,1,2,n_o}$. Inspection of Fig. 5 shows that the boundary of this chain interacts with the classical open string chain $\mathcal{V}_o$, the chain $\mathcal{V}_1$, and the classical closed string three-vertex $\mathcal{V}_{0,3}$ as follows:

4. $\mathcal{V}_2$ 由带有 $n_c = 2$ 和 $n_o \geq 0$ 的圆盘构成。此处为 $\mathcal{V}_2 = \sum_{n_o=0}^{\infty} (g_s)^{1+\frac{1}{2}n_o} \mathcal{V}_{0,1,2,n_o}$ 。观察图 5 可知，该链的边界与经典开弦链 $\mathcal{V}_o$ 、链 $\mathcal{V}_1$ 以及经典闭弦三顶点 $\mathcal{V}_{0,3}$ 的相互作用如下:

$$\partial\mathcal{V}_2 + \{\mathcal{V}_o, \mathcal{V}_2\}_o + \frac{1}{2}\{\mathcal{V}_1, \mathcal{V}_1\}_o + \left\{\mathcal{V}_1, \mathcal{V}_{0,3}\right\}_c = 0. \tag{462}$$

The recursion relations (461) and (462) can be used to discuss global symmetries of classical open string field theory generated by closed string states in the BRST cohomology [9]. An example is Poincare transformations. Such symmetries were first noted and explored by Hata and Nojiri in covariantized light-cone string field theories [111].

递推关系 (461) 和 (462) 可用于讨论 BRST 上同调中闭弦态生成的经典开弦场论整体对称性 [9], 庞加莱变换就是一个典型例子。Hata 和 Nojiri 最早在协变光锥弦场论中注意并研究了这类对称性 [111]。

5. Define now a chain $\overline{\mathcal{V}}$ of disks with all allowed numbers of open and closed punctures:

5. 现在定义一条包含所有允许数量开孔和闭孔的圆盘链 $\overline{\mathcal{V}}$:

$$\overline{\mathcal{V}} \equiv \mathcal{V}_o + \sum_{n=1}^{\infty} \mathcal{V}_n, \ \mathcal{V}_n \equiv \sum_{n_o=0}^{\infty} (g_s)^{n+\frac{1}{2}n_o-1} \mathcal{V}_{0,1,n,n_o}. \tag{463}$$

Note that the definition of $\mathcal{V}_n$ reproduces the earlier definitions for $n = 1, 2$. This time, we find that the boundary of this chain mixes with the classical closed string chain $\widehat{\mathcal{V}}_c$:

注意 $\mathcal{V}_n$ 的定义复现了 $n = 1, 2$ 的早期定义。此次我们发现，该链的边界与经典闭弦链 $\widehat{\mathcal{V}}_c$ 混合:

$$\partial\overline{\mathcal{V}} + \frac{1}{2}\{\overline{\mathcal{V}}, \overline{\mathcal{V}}\}_o + \left\{\overline{\mathcal{V}}, \widehat{\mathcal{V}}_c\right\}_c = 0, \tag{464}$$

This relation allows the formulation of a classical open string field theory in a nontrivial closed string background. The action is written with the familiar kinetic term for the open string field plus all the interactions implicit in $\overline{\mathcal{V}}$ with a closed string field Ψ_0 . The action has open string gauge invariance if Ψ_0 satisfies the classical closed string equations of motion [9].

该关系允许我们构造非平凡闭弦背景下的经典开弦场论。其作用量由我们熟悉的开弦场动能项，加上 $\overline{\mathcal{V}}$ 中隐含的所有与闭弦场 Ψ_0 的相互作用构成。当 Ψ_0 满足经典闭弦运动方程时，该作用量具有开弦规范不变性 [9]。

6. A chain $\widetilde{\mathcal{V}}$ that includes all genus zero surfaces with all numbers of boundary components ($b = 0, 1, \cdots$) and all allowed numbers of open and closed string punctures:

6. 链 $\widetilde{\mathcal{V}}$ 包含所有零亏格曲面，这些曲面带有任意数量的边界分量 ($b = 0, 1, \cdots$)，以及所有允许数量的开弦孔和闭弦孔:

$$\widetilde{\mathcal{V}} = \sum_{n_c, b, n_o} (g_s)^{n_c+b+\frac{1}{2}n_o-2} \mathcal{V}_{0,b,n_c,n_o}. \tag{465}$$

This chain, all by itself, satisfies a simple recursion relation:

该链自身就满足一个简单的递推关系:

$$\partial\widetilde{\mathcal{V}} + \frac{1}{2}\{\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}\} + \Delta_o'\widetilde{\mathcal{V}} = 0 \tag{466}$$

Here the antibracket is the full one, including the closed and open contributions. Note the action of Δ_o' in which open strings on the same boundary component are glued together. The other operator Δ_o does not feature as it increases the genus. This recursion relation has been recently shown to be relevant to the description of the physics of N D-branes in the large N limit [105,112].

此处的反括号是包含开、闭贡献的完整反括号。注意 Δ_o' 的作用是将同一边界分量上的开弦粘合在一起。另一个算符 Δ_o 不参与其中，因为它会提高亏格。近来已有研究表明，该递推关系与大 N 极限下 N D 膜的物理描述相关 [105,112]。

The recursive approach to the construction of string vertices described in the existence proof in this section can be implemented for practical computations involving low-dimensional moduli spaces. One particularly useful set of vertices arises when the starting point, the vertices with no moduli, uses local coordinates at the punctures that are related to the global coordinates on the sphere or the upper half plane via SL(2, C) or SL(2, R) maps (projective maps). In this case, the construction of $\{\mathcal{V},\mathcal{V}\}$ and $\Delta\mathcal{V}$ can be done by solving a set of algebraic equations (as opposed to differential equations for more general maps), and we can systematically proceed to construct higher-order vertices. Examples of such constructions and their practical applications can be found in [87,113-115]. As of now, however, there is no uniform description of all the $\mathcal{V}$ ’s with projective local coordinates even at genus zero. We shall now describe a few approaches that give uniform description of all the $\mathcal{V}$ ’s.

本节存在性证明中描述的弦顶点构造递归方法可用于涉及低维模空间的实际计算。当构造起点 (零模顶点) 所用的刺点局部坐标通过 SL(2, C) 或 SL(2, R) 变换 (投影变换) 与球面或上半平面的整体坐标相关联时，就能得到一组特别有用的顶点。在这种情况下，$\{\mathcal{V},\mathcal{V}\}$ 和 $\Delta\mathcal{V}$ 的构造可通过求解一组代数方程完成 (区别于更一般变换所需的微分方程)，我们还能系统地逐步构造高阶顶点。这类构造的示例及其实用应用可见文献 [87,113-115]。但截至目前，即使在零亏格情况下，也尚未对所有采用投影局部坐标的 $\mathcal{V}$ 给出统一描述。我们接下来将介绍几种能给出所有 $\mathcal{V}$ 统一描述的方法。

Minimal Area String Vertices: Witten Vertex and Closed String Polyhedra

最小面积弦顶点: 威滕顶点与闭弦多面体

We now turn to the construction of string vertices. In this subsection, we shall describe the construction based on the minimal area metric, focusing on the bosonic string theory. The most elementary vertex is the tree-level three point function. For open strings, the vertex is a disk with three punctures on the boundary. It is instructive to compare this with closed strings for which the vertex is a sphere with three punctures. For the vertices that will be constructed in this subsection, the two are related. If we view the closed string vertex as a surface with three punctures located on the equatorial circle, the open string vertex can be defined by cutting the surface along that circle and keeping just, say, the northern hemisphere. This is now a disk with three boundary punctures. The local coordinates at the punctures are inherited from the local coordinates at the punctures of the closed string vertex.

我们现在转向弦顶点的构造。在本小节中，我们将描述基于最小面积度规的构造，聚焦于玻色弦理论。最基本的顶点是树级三点顶点。对于开弦，该顶点是边界上带有三个孔的圆盘。将其与闭弦对比很有启发性: 闭弦的三顶点是带有三个孔的球面。本小节将要构造的两类顶点存在关联: 如果我们把闭弦顶点看作三个孔都位于赤道圆上的曲面，那么开弦顶点可以通过沿赤道圆切割曲面，只保留 (例如) 北半球得到，切割后就得到了一个在边界带有三个孔的圆盘。孔处的局部坐标继承自闭弦顶点孔处的局部坐标。

We begin with the simplest case: bosonic open string field theory. The vertex constructed by Witten is a special case of the general class of the three open string vertices that we shall consider. As remarked earlier, Witten’s open string field theory is cubic, i.e., this theory only needs a three-string vertex satisfying an associativity condition. We shall first describe this construction. The surface can be built by gluing together three half disks-the coordinate half-disks or patches. These half disks $\{|w_i| \leq 1, \operatorname{Im} w_i \geq 0\}$, with $i = 1, 2, 3$, can be thought as the world-sheets of the three strings and are shown in the Fig. 6. Points $w_i = 0$ are the punctures. Boundaries $|w_i| = 1$ of the three disks are glued as indicated by the coloring of the figure, and according to identifications

我们从最简单的情况开始: 玻色开弦场论。威滕构造的顶点是我们将要讨论的一类广义三开弦顶点的特殊情况。如前所述，威滕的开弦场论是三次方型的，即该理论仅需要一个满足结合性条件的三弦顶点。我们首先来描述这个构造。该世界面可以通过拼接三个半圆盘得到——也就是坐标半圆盘或坐标片。这三个半圆盘 $\{|w_i| \leq 1, \mathrm{Im}\, w_i \geq 0\}$， 在满足 $i = 1, 2, 3$ 的情况下， 可以看作是三根弦的世界面，如图 6 所示。点 $w_i = 0$ 是孔。三个圆盘的边界 $|w_i| = 1$ 按照图中的着色标记拼接，并根据等同关系粘合

$$\begin{aligned} w_1 w_2 &= -1, \quad \text{for } |w_1| = 1, \mathrm{Re}\, w_1 \leq 0, \\ w_2 w_3 &= -1, \quad \text{for } |w_2| = 1, \mathrm{Re}\, w_2 \leq 0, \\ w_3 w_1 &= -1, \quad \text{for } |w_3| = 1, \mathrm{Re}\, w_3 \leq 0. \end{aligned} \tag{467}$$

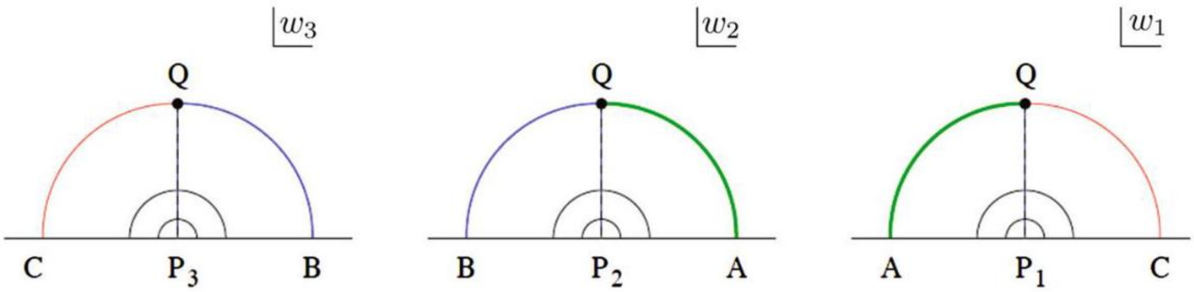


Fig. 6 Three half disks representing open string world-sheets. The boundaries are glued as indicated by the color coding, resulting in a disk with three boundary punctures

图 6 代表开弦世界面的三个半圆盘。边界按颜色标注规则粘合，最终得到一个带有三个边界孔的圆盘

Note that the common interaction point Q , at $w_i = i$ for all three disks, is the midpoint of each ”open string” $|w_i| = 1$, $\mathrm{Im}\, w_i \geq 0$. The gluing prescription identifies the left half of string one with the right half of string two, the left half of string two with the right half of string three, and the left half of string three with the right half of string one, forming a single disk with its boundary comprised by the union of the horizontal boundaries of the three disks.

注意对三个圆盘而言共同的相互作用点 Q， 位于 $w_i = i$， 是每根“开弦”$|w_i| = 1$， $\mathrm{Im}\, w_i \geq 0$ 的中点。粘合规则将弦 1 的左半部分与弦 2 的右半部分等同，弦 2 的左半部分与弦 3 的右半部分等同，弦 3 的左半部分与弦 1 的右半部分等同，最终形成一个单一圆盘，其边界由三个半圆盘的水平边界合并而成。

It is useful to visualize the three open string vertex as a surface with a metric. For this note that we have, on the glued surface, a Strebel quadratic differential φ takes the form

将三开弦顶点视为带有度规的曲面有助于直观理解，需要注意的是，粘合后的曲面上存在一个 Strebel 二次微分 φ ，其形式为

$$\varphi = \phi(w_i)\,dw_i^2 = -\frac{1}{w_i^2}dw_i^2, \tag{468}$$

holding on each of the three coordinate patches. It is crucial that φ is defined globally because local expressions above are consistent with the gluing conditions $w_i w_{i+1} = -1$. Horizontal trajectories, the lines along which φ is real and positive, represent open strings—one sees that those are the sets $w_i = re^{i\theta}$ with $r \leq 1$ fixed and $\theta \in [0, \pi]$, so that $\varphi = (d\theta)^2$. In particular, for $r = 1$, these are open strings that are being glued together. The quadratic differential has second-order poles at the punctures $w_i = 0$, as can be seen directly from the above expression. The quadratic differential has a zero at Q , where three half-neighborhoods of the point i on the disks are glued. Indeed, a well-defined coordinate z vanishing at Q takes the form $z = (w_i - i)^{2/3}$. Therefore, near $w_i = i$, we have $\varphi \simeq dw_i^2 \sim z dz^2$, making the zero at $z = 0$ manifest.

该表达式在三个坐标补丁上分别成立。关键在于 φ 是整体良定义的，因为上述局部表达式与粘合条件 $w_i w_{i+1} = -1$ 自洽。水平轨道，也就是 φ 为正实数的曲线，代表开弦——可以看出这些轨道是 $w_i = re^{i\theta}$ 固定 $r \leq 1$ 且满足 $\theta \in [0, \pi]$ 的集合，因此得到 $\varphi = (d\theta)^2$ 。特别地，对 $r = 1$ 而言，这些就是被粘合在一起的开弦。该二次微分在孔 $w_i = 0$ 处存在二阶极点，这可以从上述表达式直接看出。二次微分在 Q 处存在一个零点，三个圆盘上点 i 的三个半邻域就是在这里粘合的。确实，一个在 Q 处消失的良定义坐标 z 形式为 $z = (w_i - i)^{2/3}$ 。因此，在 $w_i = i$ 附近我们有 $\varphi \simeq dw_i^2 \sim z dz^2$ ，这清晰地显示了 $z = 0$ 处的零点。

A quadratic differential has a transformation law $\phi(z)\,dz^2 = \phi(w)\,dw^2$ under analytic changes of coordinates that implies the existence of a well-defined conformal metric

二次微分在解析坐标变换下满足变换规律 $\phi(z)\,dz^2 = \phi(w)\,dw^2$ ，这表明存在定义良好的共形度量

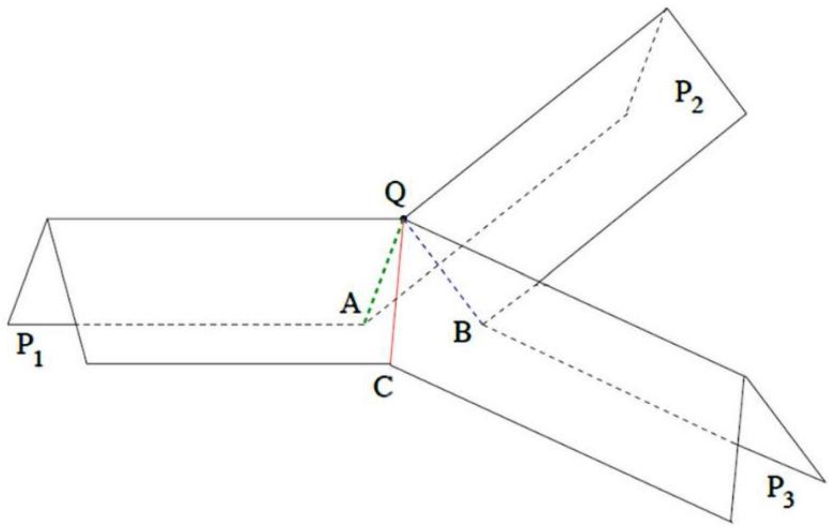
$$ds^2 = |\phi(w_i)|\,|dw_i|^2. \tag{469}$$

If we write $w_i = e^{-T}e^{i\theta}$, the half disk corresponds to $T \in [0, \infty]$ and $\theta \in [0, \pi]$ with flat metric $ds^2 = dT^2 + d\theta^2$. This is a semi-infinite strip of width π . The string vertex can therefore be viewed as the joining of three such strips, with half-string identifications at the edges $T = 0, \theta \in [0, \pi]$. This requires a bit of folding of the three strips along the locus of the open string midpoints. This is shown in Fig. 7. We will later explain that this is a minimal area metric on this surface under the condition that any open curve with boundary endpoints and homotopic to a puncture has length at least π . This picture of the three strips makes the associativity of the string vertex manifest. One can view a product $A \star B$ of string fields as the insertion of the state A at the infinite end of one of the vertex strips and B at the infinite end of the strip next to the first. The product is read at the last strip. For nested products, we simply add more strips in the same fashion, gluing half strings. Due to the associativity of the $\star$ -product, we have $\{\mathcal{V}_3^o, \mathcal{V}_3^o\} = 0$, implying that there is no need for higher vertices in the classical theory. It also seems clear that $\Delta \mathcal{V}_3^o \neq 0$, the left-hand side being a singular surface. This means that the $\mathcal{V}_3^o$ vertex is problematic at the quantum level [61].

若我们写出 $w_i = e^{-T}e^{i\theta}$，半圆盘对应 $T \in [0,\infty]$ 和 $\theta \in [0,\pi]$，带有平坦度量 $ds^2 = dT^2 + d\theta^2$。这是一个宽度为 π 的半无限带。因此弦顶点可以看作三个这样的带拼接而成，在边 $T = 0, \theta \in [0,\pi]$ 处对半弦做等同认同。这需要将三个带沿着开弦中点的轨迹进行少许折叠。如图 7 所示。我们后续会说明，在任何端点在边界上且同伦于一个 puncture 的开曲线长度至少为 π 的条件下，这是该曲面的极小面积度量。这个三带的图景直观体现了弦顶点的结合性。我们可以将弦场的乘积 $A \star B$ 看作: 把态 A 插入其中一个顶点带的无穷远端，把态 B 插入相邻带的无穷远端，乘积的结果从最后一个带读出。对于嵌套乘积，我们只需以相同方式添加更多带，粘合半弦即可。由于 $\star$ 乘积满足结合性，我们得到 $\{\mathcal{V}_3^o, \mathcal{V}_3^o\} = 0$，这意味着经典理论中不需要更高阶顶点。同时显然有 $\Delta \mathcal{V}_3^o \neq 0$，其左侧是一个奇异曲面。这说明 $\mathcal{V}_3^o$ 顶点在量子层面存在问题 [61]。

Fig. 7 The associative open string vertex, with the half disks mapped into three semi-infinite strips

图 7 结合性开弦顶点，半圆盘被映射为三个半无限带



For any explicit calculation of open string couplings, we need the vertex described in terms of a single disk $|w| \le 1$ with the three half-disks embedded inside it. This can be achieved with a sequence of conformal maps, with each half disk going into a 120° wedge of the unit disk. To map w_i to this wedge, we first use an $SL(2,\mathbb{C})$ transformation

要对开弦耦合做任何显式计算，我们需要用单个圆盘 $|w| \le 1$ (其中嵌入三个半圆盘) 来描述顶点。这可以通过一系列共形变换实现: 每个半圆盘被映射到单位圆盘的一个 120° 楔形区。为了将 w_i 映射到这个楔形区，我们首先使用一个 $SL(2,\mathbb{C})$ 变换

$$h(u) = \frac{1+iu}{1-iu} \tag{470}$$

This maps the unit upper half disk $\{|u| < 1, \operatorname{Im} u \ge 0\}$ to the "right" half disk $\{|h| \le 1, \operatorname{Re} h \ge 0\}$. It is now simple to apply the h map to the w_i half disks to create the correct wedges by the power map $h^{2/3}$ and then to place the wedges properly in the w disk by global rotations. One then gets that the maps

它将单位上半圆盘 $\{|u| < 1, \text{Im}\, u \geq 0\}$ 映射为“右侧”半圆盘 $\{|h| \leq 1$，满足 $\text{Re}\, h \geq 0\}$。接下来很容易通过幂映射 $h^{2/3}$ 对 w_i 个半圆盘应用 h 映射得到正确的楔形区，再通过整体旋转将楔形区正确放置在 w 圆盘中。最终得到映射

$$w(w_1) = e^{2\pi i/3}(h(w_1))^{2/3},\; w(w_2) = (h(w_2))^{2/3},\; w(w_3) = e^{-2\pi i/3}(h(w_3))^{2/3},$$

(471)

which indeed take the half disks into wedges fitting the w unit disk as shown in Fig. 8. A final map to the upper half plane is also quite useful. This is done by composing the above functions with the map $z = h^{-1}(w) = -i\frac{w-1}{w+1}$ taking the unit w disk to the upper half plane, with the punctures on the real axis. We thus get

它确实如 Fig.8 所示将半圆盘映射为适配 w 单位圆盘的楔形区。到上半平面的最终映射也十分有用。这可以通过将上述函数与映射 $z = h^{-1}(w) = -i\frac{w-1}{w+1}$ 复合得到: 映射 $z = h^{-1}(w) = -i\frac{w-1}{w+1}$ 将单位 w 圆盘映射到上半平面，puncture 落在实轴上。于是我们得到

$$z = f_1(w_1) = h^{-1}(w(w_1)) = \sqrt{3} + \frac{8}{3}w_1 + \frac{16}{9}\sqrt{3}w_1^2 + \frac{248}{81}w_1^3 + \mathcal{O}\left(w_1^4\right),$$

$$z = f_2(w_2) = h^{-1}(w(w_2)) = \frac{2}{3}w_2 - \frac{10}{81}w_2^3 + \mathcal{O}\left(w_2^5\right),$$

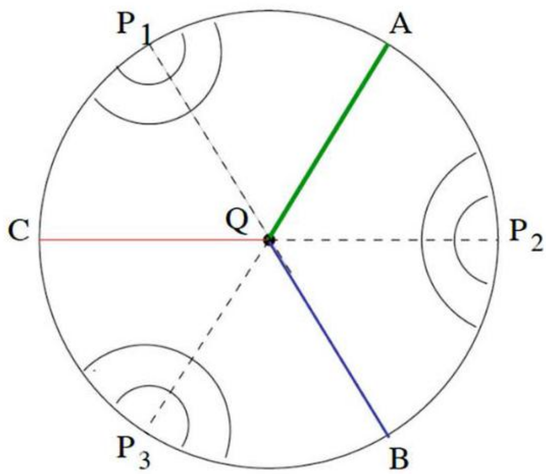
$$z = f_3(w_3) = h^{-1}(w(w_3)) = -\sqrt{3} + \frac{8}{3}w_3 - \frac{16}{9}\sqrt{3}w_3^2 + \frac{248}{81}w_3^3 + \mathcal{O}\left(w_3^4\right).$$

(472)

We have included above the power series expansion of the local coordinates. These are needed for explicit calculations. The picture of the vertex in the z upper-half plane is shown below. The image of the three half disks fill the upper half plane. The string midpoint Q now appears at $z = i$ (see Fig. 9).

我们已经在上面给出了局部坐标的幂级数展开。显式计算需要用到这些展开。顶点在 z 上半平面的图像如下图所示。三个半圆盘的像覆盖了整个上半平面。弦中点 Q 现在出现在 $z = i$ 处 (见图 9)。

Fig. 8 The associative open string vertex presented globally on a $|w| \leq 1$ disk with punctures at P_1, P_2, P_3 and common mid-point Q



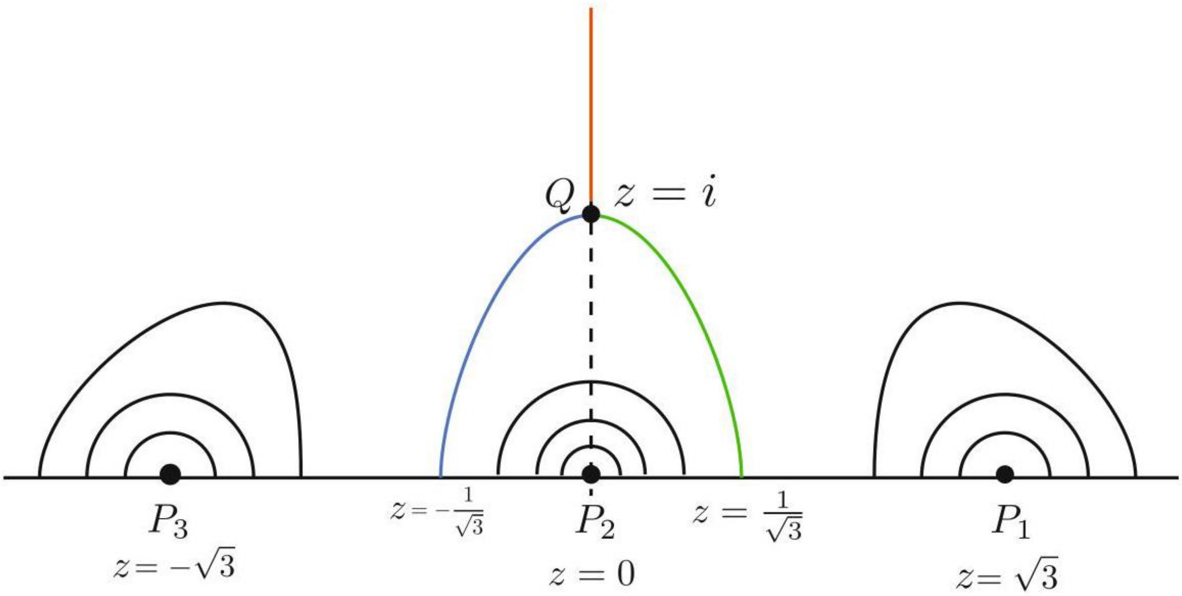


Fig. 9 The associative open string vertex represented on the upper-half z -plane. The punctures now lie on the real line, and the three half disks cover the full upper-half plane

图 9 上半 z 平面上表示的结合开弦顶点。刺点都位于实轴上，三个半圆盘覆盖了整个上半平面

For closed strings, the three-string vertex is simply the open string vertex, as described by the above z upper-half plane, extended to the full z plane. The coordinates at the punctures are the same as well. One can visualize the closed string vertex as the result of gluing three semi-infinite cylinders of circumference 2π across the open surfaces (Fig. 10). The gluing represents the doubling of the gluing of three semi-infinite strips for the open string vertex. The resulting picture is as shown below. As before, we have a Strebel quadratic differential on this surface, with second-order poles at the three punctures. This time, there are two zeroes of the quadratic differential, and these are the two heavy dots in the figure, representing the location where the three boundaries of the semi-infinite cylinders meet.

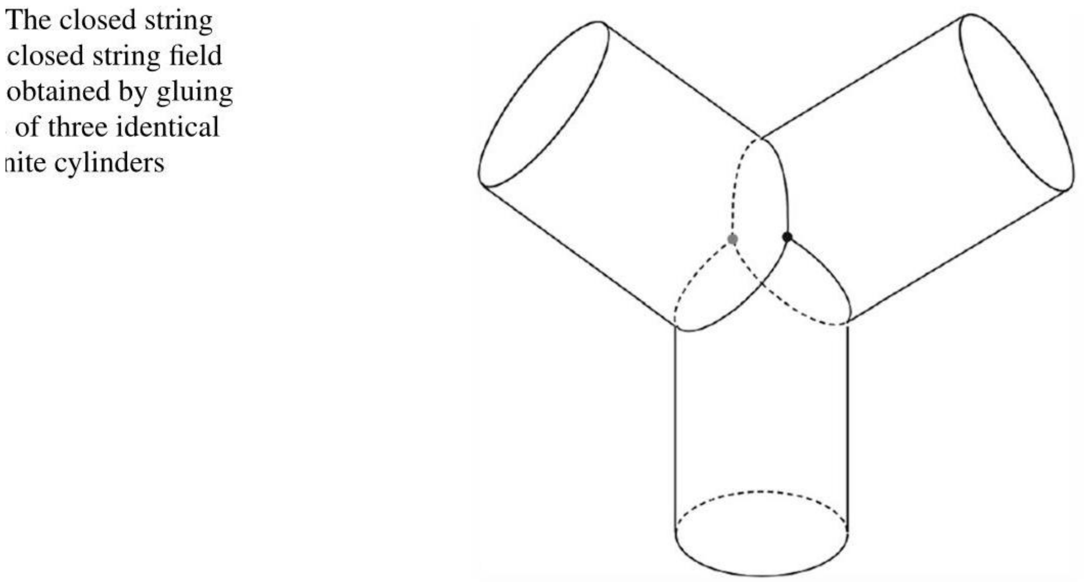
对于闭弦，三弦顶点就是上述 z 上半平面描述的开弦顶点延拓到整个 z 平面得到的。刺点处的坐标也完全相同。可以将闭弦想象为把三个周长为 2π 的半无限圆柱粘贴在开曲面上得到的结果 (图 10)。这个粘贴对应于开弦顶点三个半无限条带粘贴的加倍。所得图像如下图所示。和之前一样，这个曲面上存在一个 Strebel 二次微分，在三个刺点处有二阶极点。本次二次微分有两个零点，即图中的两个黑点，它们代表三个半无限圆柱的边界相交的位置。

As mentioned before, we have $\partial\mathcal{V}_{0,4} = -\frac{1}{2}\{\mathcal{V}_{0,3},\mathcal{V}_{0,3}\}$, and we must now understand what is the four-string vertex $\mathcal{V}_{0,4}$. Physically, the necessity for $\mathcal{V}_{0,4}$ arises because the three Feynman diagrams, formed using two copies of the vertex, do not provide a full cover of the moduli space of four-punctured spheres. These three diagrams are shown in Fig. 11. The antibracket actually constructs the boundary of the Feynman regions covered by the three diagrams. If we place three of the labeled punctures at0,1, and ∞ , the Feynman regions, shown in white in Fig. 12, are disks around0,1, and ∞ , with boundaries $\mathcal{B}_1,\mathcal{B}_2$, and $\mathcal{B}_3$. Vertex $\mathcal{V}_{0,4}$ represents the shaded region of the moduli space. The darker shading shows a fundamental domain, such that the action of the $\mathrm{SL}(2,\mathbb{C})$ transformations that permute three fixed punctures at0,1, and ∞ produces the full shaded region. The local coordinates at the punctures are known over the $\mathcal{B}_i$ sets, and the local coordinates over $\mathcal{V}_{0,4}$ must match these. We shall now describe a natural way to define the local coordinates over $\mathcal{V}_{0,4}$ and give a simple parameterization of the shaded region. Both of these issues can be addressed by considering a theorem of Strebel, showing the existence and uniqueness of a certain type of quadratic differentials. The relevant version of the theorem is easily understood [117, 118].

如前所述，我们已经有了 $\partial\mathcal{V}_{0,4} = -\frac{1}{2}\{\mathcal{V}_{0,3},\mathcal{V}_{0,3}\}$，现在必须理解四弦顶点 $\mathcal{V}_{0,4}$ 是什么。从物理上看，$\mathcal{V}_{0,4}$ 的必要性源于，使用两个三弦顶点构造出的三个费曼图无法完全覆盖四 punctured 球面的模空间。这三个图如图 11 所示。反括号实际上构造出了这三个图覆盖的费曼区域的边界。如果我们把三个标记刺点放在 0、1 和 ∞ 处，那么图 12 中白色区域所示的费曼区域就是 0、1 和 ∞ 周围的圆盘，边界为 $\mathcal{B}_1,\mathcal{B}_2$ 和 $\mathcal{B}_3$ 。顶点 $\mathcal{V}_{0,4}$ 对应模空间的阴影区域。深色阴影是一个基本域，对置换 0、1、∞ 这三个固定刺点的 $\mathrm{SL}(2,\mathbb{C})$ 变换而言，该基本域在这些变换作用下就得到整个阴影区域。刺点处的局部坐标在 $\mathcal{B}_i$ 集合上是已知的，$\mathcal{V}_{0,4}$ 上的局部坐标必须与这些坐标匹配。我们现在将描述一种在 $\mathcal{V}_{0,4}$ 上定义局部坐标的自然方法，并给出阴影区域的简单参数化。这两个问题都可以通过考虑 Strebel 的一个定理来解决，该定理证明了某类二次微分的存在性和唯一性。该定理的相关表述很容易理解 [117, 118]。

semi-infinite cylinders

半无限圆柱



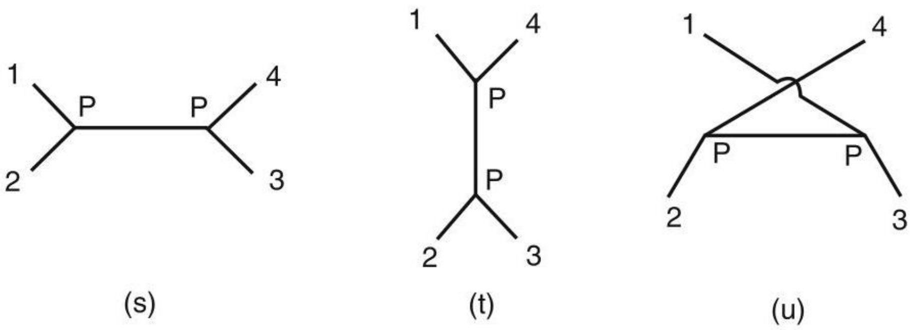


Fig. 11 Three Feynman diagrams for four-string scattering, indicated as s, t, u channels

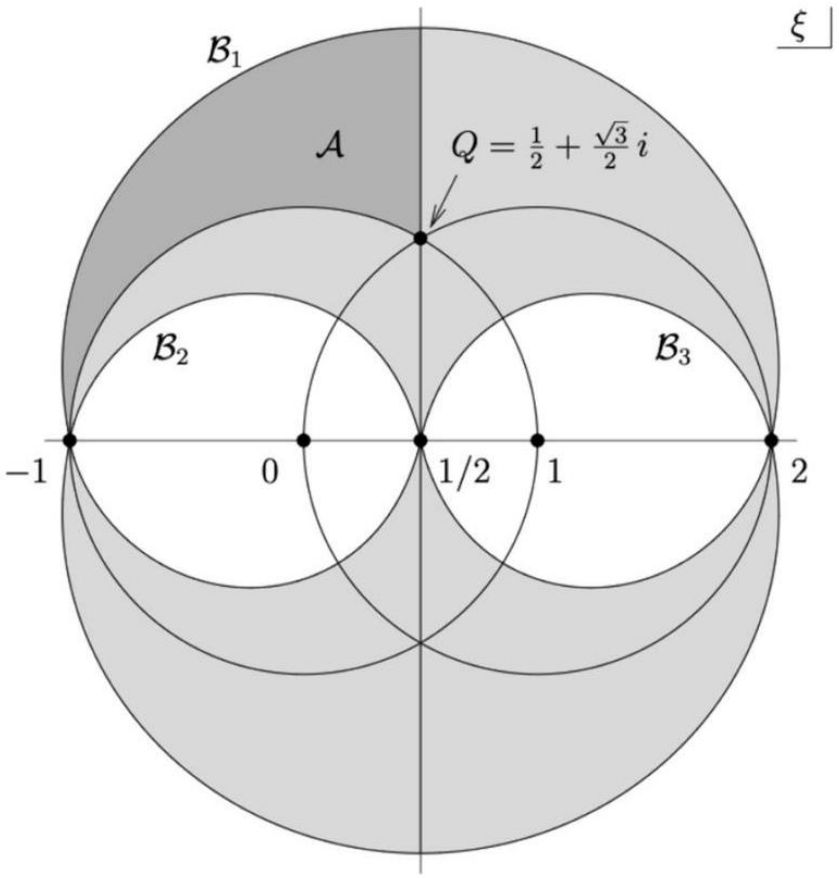
图 11 四弦散射的三个费曼图，标记为 s, t, u 道

Any n-punctured Riemann surface can be built uniquely with n identical semi-infinite cylinders, by gluing their boundaries with some set of isometric identifications.

任意 n 刺点黎曼曲面都可以通过 n 个相同的半无限圆柱，沿其边界做等距粘合唯一构造出来。

Fig. 12 As calculated and plotted by Moeller [116], the shaded region on the ξ plane represents the string vertex $\mathcal{V}_{0,4}$. Boundaries $\mathcal{B}_i$ are obtained from the Feynman region as the propagator length goes to zero

图 12 由 Moeller 计算并绘制 [116]，ξ 平面上的阴影区域代表弦顶点 $\mathcal{V}_{0,4}$ 。边界 $\mathcal{B}_i$ 是传播子长度趋于零时从费曼区域得到的



For convenience, we will assume that the cylinders have circumference 2π . The resulting surface has a Strebel quadratic differential, which is equal to $\varphi = -\frac{1}{w_i^2} dw_i^2$ on each semi-infinite cylinder, described as the disk $|w_i| \leq 1$, punctured at $w_i = 0$. We can choose the w_i ' s as the local coordinates as the punctures, producing a section of $\widehat{\mathcal{P}}_{g,n}$. We shall see later that a part of of this section for genus zero $\widehat{\mathcal{P}}_{0,n}$ can be identified as the classical vertices $\mathcal{V}_{0,n}$. Note that, in fact, vertex $\mathcal{V}_{0,3}$ was constructed in this fashion. Note also that the boundaries of the cylinders, once glued, define a critical graph; this graph has n faces, one for each cylinder, a number of vertices where φ vanishes and a number of edges, each connecting a pair of vertices. One can see from Fig. 10 that for the three string vertices, the critical graph has three faces, three edges, and two vertices. If the Strebel differential is known, the local coordinates at the punctures are clearly fixed.

为方便起见，我们假设柱面的周长为 2π 。所得曲面上存在一个 Strebel 二次微分，它在每个半无限柱面上等于 $\varphi = -\frac{1}{w_i^2} dw_i^2$ ，该曲面可描述为在 $w_i = 0$ 处打孔的圆盘 $|w_i| \leq 1$ 。我们可以选择 w_i 作为打孔处的局部坐标，从而得到 $\widehat{\mathcal{P}}_{g,n}$ 的一个截面。后文我们会看到，零亏格 $\widehat{\mathcal{P}}_{0,n}$ 对应的该截面的一部分可等同于经典顶点 $\mathcal{V}_{0,n}$ 。注意，实际上顶点 $\mathcal{V}_{0,3}$ 正是以这种方式构造的。另需注意，柱面边界粘合后会定义一个临界图；该图有 n 个面 (每个柱面对应一个面)、若干个 φ 为零的顶点，以及若干条连接顶点对的边。从图 10 可以看出，三弦顶点的临界图有三个面、三条边和两个顶点。若确定了 Strebel 微分，打孔处的局部坐标就显然被固定了。

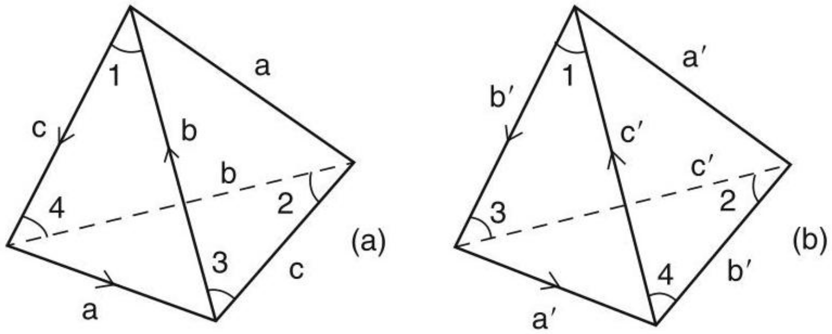
For $\mathcal{V}_{0,4}$, by Strebel' s theorem, the surfaces are constructed with four cylinders. The graph indicating the identifications must have four faces. The zeroes of the quadratic differential are trivalent vertices in this graph. These zeroes are points of localized negative curvature, elsewhere the metric is flat, though each puncture contributes to the total accounting of curvature in the Gauss-Bonnet formula. It turns out that the quadratic differential generically has four zeroes, and thus the critical graph has four vertices. A little thought reveals that the critical graph of Strebel differentials for $\mathcal{V}_{0,4}$ defines a tetrahedron. The graph is determined if we know the lengths of the edges of the tetrahedron. We have six edges and four length conditions, since the edges on each face must add up to 2π . This means two real parameters determine the unique quadratic differential—a fact consistent with $\mathcal{M}_{0,4}$ being a space of complex dimension one. We can work with three real parameters a, b, c that, representing the lengths of the three edges on a face, are all non-negative and add up to 2π : $a+b+c=2\pi$. It is simple to see that all four faces have edges of length a, b , and c . Figure 13 shows the tetrahedra, with the two possible inequivalent ways of labeling the faces of the critical graph. For the second configuration, we use parameters a', b', c' satisfying $a'+b'+c'=2\pi$. One must now find the region of $\{a, b, c\}$ parameter space for $\mathcal{V}_{0,4}$. The full parameter space $a+b+c=a'+b'+c'=2\pi$ would produce all surfaces in $\mathcal{M}_{0,4}$, so we must determine the restricted tetrahedra that correspond to $\mathcal{V}_{0,4}$. It is not difficult to track the tetrahedra obtained from $\{\mathcal{V}_{0,3}, \mathcal{V}_{0,3}\}$, thus determining the boundary of $\mathcal{V}_{0,4}$. In fact, since that tetrahedron comes from the gluing of two $\mathcal{V}_{0,3}$ and twisting, one finds that one length parameter becomes equal to π . It then becomes clear that the regions that define $\mathcal{V}_{0,4}$ are

对于 $\mathcal{V}_{0,4}$, 根据 Strebel 定理, 曲面由四个柱面构造而成, 标识对应图必须有四个面。二次微分的零点是该图中的三价顶点，这些零点是局域负曲率点，除此之外度量都是平坦的——不过每个打孔都会在高斯-博内公式的总曲率计算中贡献曲率。结果表明，二次微分一般有四个零点，因此临界图有四个顶点。稍加思考可知，$\mathcal{V}_{0,4}$ 对应的 Strebel 微分的临界图定义了一个四面体。只要知道四面体各边的长度就能确定这个图。我们有六条边和四个长度约束，因为每个面的边长之和必须等于 2π 。这意味着仅需两个实参数就能确定唯一的二次微分——这与 $\mathcal{M}_{0,4}$ 是复一维空间的结论一致。我们可以使用三个实参数 a, b, c ，它们分别代表一个面上三条边的长度，均非负且总和为 2π ：$a+b+c=2\pi$ 。不难看出，四个面的边长分别为 a, b 和 c 。图 13 展示了四面体，包含两种不等价的临界图面标记方式。对于第二种构型，我们使用满足 $a'+b'+c'=2\pi$ 的参数 a', b', c' 。现在需要找出 $\mathcal{V}_{0,4}$ 对应的 $\{a, b, c\}$ 参数空间区域。全参数空间 $a+b+c=a'+b'+c'=2\pi$ 会给出 $\mathcal{M}_{0,4}$ 中的所有曲面，因此我们需要确定对应 $\mathcal{V}_{0,4}$ 的受限四面体。不难追踪由 $\{\mathcal{V}_{0,3}, \mathcal{V}_{0,3}\}$ 得到的四面体，从而确定 $\mathcal{V}_{0,4}$ 的边界。实际上，由于该四面体来自两个 $\mathcal{V}_{0,3}$ 的粘合与扭转，可发现一个长度参数等于 π 。由此可以明确，定义 $\mathcal{V}_{0,4}$ 的区域为

$$a, b, c \leq \pi, \ a', b', c' \leq \pi. \tag{473}$$

Fig. 13 The edges of the four semi-infinite cylinders are glued to each other following the overapp patterns of the above tetrahedra, with one cylinder attached to each face

图 13 四个半无限圆柱的边缘按照上述四面体的重叠模式相互粘合，每个面连接一个圆柱



The region is an interior triangle on the a, b, c parameter space, shown in purple in Fig. 14. The other three triangles represent the Feynman regions. A simple change of viewpoint allows a simple characterization of the restricted tetrahedra. Consider closed paths, not homotopic to the punctures on the critical graph. On the left tetrahedron of Fig. 13, a path surrounding faces 1 and 2 has length $2(b+c)$. The condition $a \leq \pi$ implies $2\pi-(b+c) \leq \pi$, and thus $b+c \geq \pi$, or equivalently $2(b+c) \geq 2\pi$. The length condition makes this path longer or equal to 2π , the circumference of any cylinder. Length conditions (473) are in fact equivalent to the condition that all nontrivial closed paths on the critical graph must be longer than or equal to 2π .

该区域是 a, b, c 参数空间上的一个内三角形，在图 14 中以紫色显示。另外三个三角形代表费恩曼区域。改变视角后就能简单刻画受限四面体的特征。考虑临界图上不同于孔的非同伦闭路径。在图 13 的左侧四面体中，包围面 1 和面 2 的路径长度为 $2(b+c)$ 。条件 $a \leq \pi$ 蕴含 $2\pi-(b+c) \leq \pi$ ，进而得到 $b+c \geq \pi$ ，等价于 $2(b+c) \geq 2\pi$ 。长度条件要求该路径长度大于或等于任意圆柱的周长 2π 。实际上，长度条件 (473) 等价于: 临界图上所有非平凡闭路径的长度都必须大于或等于 2π 。

This simple condition in fact works for all $\mathcal{V}_{0,n}$ with $n \geq 4$, that is, for all classical closed string interactions. Indeed, for any fixed n , consider the graph with n faces and introduce length parameters for the gluing of the n semi-infinite cylinder boundaries. We then have [118-120]

这个简单条件实际上对所有满足 $n \geq 4$ 的 $\mathcal{V}_{0,n}$ 都成立，也就是对所有经典闭弦相互作用都成立。实际上，对任意固定的 n ，考虑拥有 n 个面的图，并为 n 个半无限圆柱边界的粘合引入长度参数，我们得到 [118-120]

The surfaces in $\mathcal{V}_{0,n}$ are those for which the critical graph has no nontrivial closed path shorter than 2π . We call such critical graphs restricted polyhedra.

$\mathcal{V}_{0,n}$ 中的曲面满足: 其临界图不存在长度小于 2π 的非平凡闭路径。我们将这类临界图称为受限多面体。

A relatively straightforward argument allows one to show that the geometric master equation is satisfied for this choice of vertices. There are two elements to that argument as one tries to establish the equality of $\{\mathcal{V}, \mathcal{V}\}$ and $\partial\mathcal{V}$. First, $\partial\mathcal{V} \subset \{\mathcal{V}, \mathcal{V}\}$ follows because the boundary of $\mathcal{V}$ is polyhedra with some closed path reaching the critical length of 2π , and if we cut along this path, we obtain two restricted polyhedra. Second, $\{\mathcal{V}, \mathcal{V}\} \subset \partial\mathcal{V}$, because when two restricted polyhedra are glued across a face, we get a restricted polyhedron-this is argued by showing that any closed path that now runs over the whole glued surface is still long enough.

我们可以通过一个相对直接的论证证明，这种顶点选择满足几何主方程。在尝试证明 $\{\mathcal{V}, \mathcal{V}\}$ 与 $\partial\mathcal{V}$ 相等时，论证包含两个部分。首先，$\partial\mathcal{V} \subset \{\mathcal{V}, \mathcal{V}\}$ 成立是因为 $\mathcal{V}$ 的边界是多面体，存在闭路径达到临界长度 2π ，沿该路径切割就能得到两个受限多面体。其次，$\{\mathcal{V}, \mathcal{V}\} \subset \partial\mathcal{V}$ 成立是因为当两个受限多面体沿一个面粘合时，得到的仍然是受限多面体——我们可以通过论证证明，新的任意闭路径即便延伸至整个粘合曲面，长度仍然满足要求。

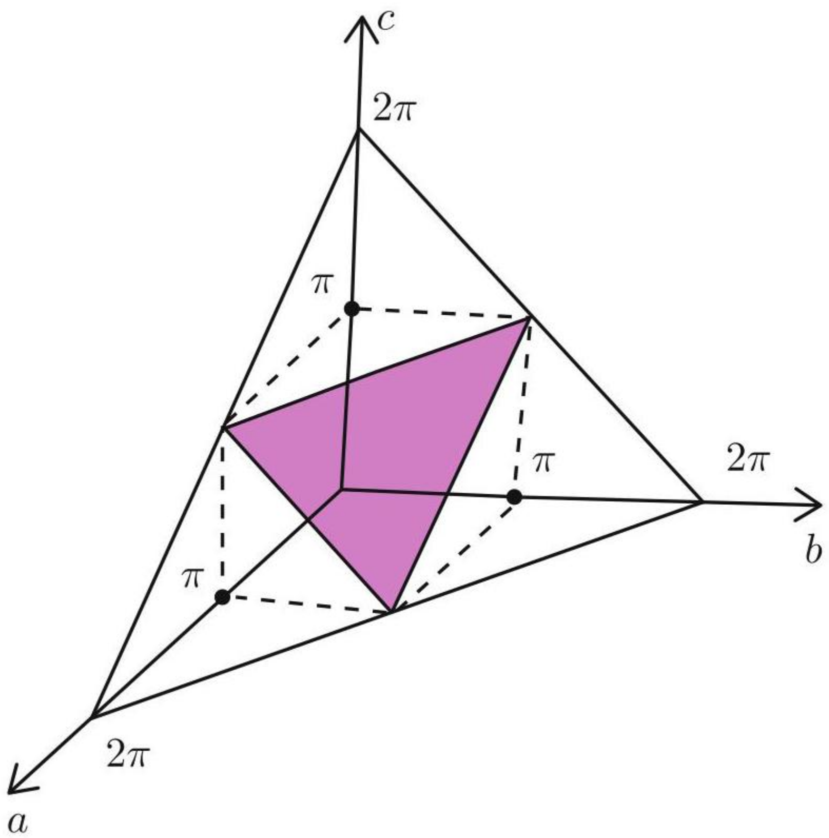


Fig. 14 In the two-dimensional parameter space $a + b + c = 2\pi$ with $a, b, c \geq 0$, the vertex region is represented by the (purple) middle triangle

图 14 在二维参数空间 $a + b + c = 2\pi$ 且满足 $a, b, c \geq 0$ 的条件下，顶点区域由中间的 (紫色) 三角形表示

While the regions $\mathcal{V}_{0,n}$ have a very elegant description, the explicit construction of the quadratic differentials is quite challenging. The cases of $\mathcal{V}_{0,4}$ and $\mathcal{V}_{0,5}$ have been considered in some detail by Moeller [116, 121]. A new approach to obtain these quadratic differentials based on machine learning has been recently introduced by Erbin and Firat [122] and could eventually help carry out computations for vertices $\mathcal{V}_{0,n}$ for much larger n . An approach based on Liouville conformal field theory to calculate quadratic differentials has been proposed in [123].

尽管区域 $\mathcal{V}_{0,n}$ 的描述非常简洁，但二次微分的显式构造相当有难度。Moeller 已经对 $\mathcal{V}_{0,4}$ 和 $\mathcal{V}_{0,5}$ 的情况做了详细研究 [116, 121]。Erbin 和 Firat 最近提出了一种基于机器学习获取这类二次微分的新方法 [122]，该方法最终有望帮助我们对大得多的 n 对应的顶点 $\mathcal{V}_{0,n}$ 开展计算。文献 [123] 还提出了一种基于刘维尔共形场论计算二次微分的方法。

In the regions of moduli spaces not included in the $\mathcal{V}$ vertices, the Strebel quadratic differential still can build the surface, but the polyhedron has nontrivial closed paths with length smaller than 2π . The string field theory does not use these quadratic differentials. Instead, these surfaces are built using lower vertices and propagators. Consequently, the associated quadratic differential φ that determines the local coordinates w_i at the punctures via $\varphi = -dw_i^2/w_i^2$ and the condition that $|w_i| = 1$ at the end of the semi-infinite cylinder is different. It is a Jenkins-Strebel quadratic differential-this means that horizontal trajectories fill the surface. In particular, the surface can contain ring domains and finite cylinders associated with propagators and foliated by horizontal trajectories. Since the propagators have circumference 2π , all nontrivial closed curves will be larger than 2π . Note that a Strebel differential is just a particular class of a Jenkins-Strebel differential.

在模空间未包含在 $\mathcal{V}$ 顶点中的区域，Strebel 二次微分仍然可以构造曲面，但多面体存在长度小于 2π 的非平凡闭路径。弦场论不使用这些二次微分，这类曲面反而可通过更低阶顶点和传播子构造。因此，通过 $\varphi=-dw_i^2/w_i^2$ 在穿刺点处确定局部坐标 w_i 、且满足半无限圆柱末端条件 $|w_i|=1$ 的关联二次微分 φ 是不同的。它是 Jenkins-Strebel 二次微分——这意味着水平轨线填满整个曲面。具体而言，曲面可以包含与传播子关联、由水平轨线叶化的环形域和有限圆柱。由于传播子的周长为 2π ，所有非平凡闭曲线的长度都将大于 2π 。注意 Strebel 微分只是 Jenkins-Strebel 微分的一个特殊类别。

In order to have a single principle that determines the full section $\mathcal{F}_{0,n}$ in $\widehat{\mathcal{P}}_{0,n}$, we turn to a minimal area problem. In fact, this minimal area problem in principle determines the quantum vertices $\mathcal{V}_{g,n}$ and full sections $\mathcal{F}_{g,n}$ in $\widehat{\mathcal{P}}_{g,n}$ [124]:

为了得到一个能确定 $\widehat{\mathcal{P}}_{0,n}$ 中全截面 $\mathcal{F}_{0,n}$ 的单一原理，我们转向极小面积问题。事实上，该极小面积问题原则上可以确定 $\widehat{\mathcal{P}}_{g,n}$ 中的量子顶点 $\mathcal{V}_{g,n}$ 和全截面 $\mathcal{F}_{g,n}$ [124]:

Minimal area problem: Given a genus g Riemann surface with $n\geq 0$ punctures ($n\geq 3$ for $g=0, n\geq 1$ for $g=1$) find the metric of minimal (reduced) area under the condition that the length of any nontrivial homotopy closed curve be greater than or equal to 2π .

极小面积问题: 给定一个亏格为 g 、带有 $n\geq 0$ 个穿刺点 ($n\geq 3$ (对应 $g=0, n\geq 1$ 、$g=1$) 的黎曼曲面，在任意非平凡同伦闭曲线长度大于等于 2π 的条件下，寻找最小 (约化) 面积的度量。

Here in this problem, a metric means a conformal metric, that is, one has a function ρ such that the length element is $ds=\rho\,|dz|$. Since the problem involves all nontrivial closed curves in a surface, it is ”modular invariant” (the set of all nontrivial closed curves is invariant under large diffeomorphisms) and is independent of the labeling of the punctures, thus implementing naturally the symmetry under the exchange of punctures. It is simple to show that any solution to a minimal area problem is unique, and this follows from the convexity of the area functional. Uniqueness is the reason we get a section in $\widehat{\mathcal{P}}_{g,n}$. In the minimal area metric, one expects semi-infinite cylinders associated with the punctures. The term ”reduced” area is used because the area of the semi-infinite cylinders is infinite and requires regularization. One uses fixed arbitrary coordinates $\widetilde{w}$ around the punctures to remove small disks $|\widetilde{w}|\leq\varepsilon$ around the puncture and subtract off a logarithmic divergence $\sim -\log\varepsilon$ in the area. Under a change of the arbitrary coordinates around the punctures, the reduced area changes by a metric independent constant. This implies that the extremal metric is regulator independent. The regulator also preserves the convexity of the area functional.

在该问题中，度量指共形度量，即存在函数 ρ 使得线元为 $ds = \rho\,|dz|$ 。由于该问题涉及曲面上所有非平凡闭曲线，它是“模不变”的 (所有非平凡闭曲线的集合在大微分同胚下不变)，且与穿刺点的标号无关，因此自然实现了穿刺点交换下的对称性。不难证明，极小面积问题的任何解都是唯一的，这源于面积泛函的凸性。唯一性是我们能在 $\widehat{\mathcal{P}}_{g,n}$ 中得到一个截面的原因。在极小面积度量中，我们预期穿刺点关联半无限圆柱。使用“约化”面积一词是因为半无限圆柱的面积是无穷大，需要正则化: 我们利用穿刺点附近固定的任意坐标 ϖ 移除穿刺点周围的小圆盘 $|\varpi| \leq \varepsilon$ ，并扣除面积中的对数发散项 $\sim -\log\varepsilon$ 。当改变穿刺点附近的任意坐标时，约化面积仅改变一个与度量无关的常数，这说明极值度量不依赖于正则化因子，该正则化也保留了面积泛函的凸性。

We can state what the vertices $\mathcal{V}_{g,n}$ are based on the expected properties of the minimal area metrics. All we need is an algorithm to decide if any surface $\sum_{g,n}$ is in $\mathcal{V}_{g,n}$. For this we use the minimal area metric on $\sum_{g,n}$ and inspect it searching for ring domains, or annuli, revealing the existence of propagators. A ring domain, or a cylinder of height greater or equal to 2π , is declared to be a propagator. $\sum_{g,n} \in \mathcal{V}_{g,n}$ if the minimal area metric has no propagators. To fix the local coordinates on $\sum_{g,n}$, we look at the semi-infinite cylinders associated with the punctures. These cylinders are well defined in that it must end; there must be a ”last” geodesic homotopic to the puncture. That last geodesic must be retracted toward the puncture by a distance π , and this retracted curve becomes the coordinate curve for this puncture. This procedure effectively adds length π stubs to the surface. Equipped with such stubs, one can see that $\Delta\mathcal{V}$ will not produce surfaces with closed curves shorter than 2π : when gluing two coordinate curves in a given surface, the stubs already provide 2π length to any closed curve that goes through the created handle. In fact, we get a solution of the master equation for any stub length greater than π . In the string field theory, the change of stub length is realized by a field redefinition. The longer the stub length, the less moduli space the Feynman diagrams produce and the more vertices produce. Changing the stub length has an interpretation as a renormalization group transformation of the action [125].

我们可以根据极小面积度量的预期性质阐述顶点 $\mathcal{V}_{g,n}$ 的定义。我们只需要一个算法来判断任意曲面 $\sum_{g,n}$ 是否属于 $\mathcal{V}_{g,n}$ 。为此，我们利用 $\sum_{g,n}$ 上的极小面积度量，对其进行检查，寻找显示传播子存在的环形区域 (即 annulus)。高度大于等于 2π 的环形区域 (也就是圆柱) 会被判定为传播子。如果极小面积度量不含传播子，就得到 $\sum_{g,n} \in \mathcal{V}_{g,n}$ 。为了确定 $\sum_{g,n}$ 上的局部坐标，我们考察与穿孔关联的半无限圆柱。这些圆柱是良定义的: 它必定有端点，一定存在一条与穿孔同伦的“最后一条”测地线。这条最后测地线需要向穿孔方向回缩距离 π ，回缩后得到的曲线就成为该穿孔的坐标曲线。这个过程实际上给曲面添加了长度为 π 的短柱。配备这种短柱后，可以发现 $\Delta\mathcal{V}$ 不会产生闭曲线长度短于 2π 的曲面: 当把给定曲面上的两条坐标曲线粘合时，短柱已经给穿过新生成手柄的任意闭曲线提供了长度 2π 。事实上，对于任意大于 π 的短柱长度，我们都能得到主方程的解。在弦场论中，短柱长度的改变是通过场重定义实现的。短柱长度越长，费曼图贡献的模空间越小，顶点贡献的模空间越大。改变短柱长度可以解释为作用量的重整化群变换 [125]。

We note that as long as we only work with classical closed string field theory, there is no need for stubs-and the coordinate curves coincide with the faces of the polyhedron. A fair amount of work has been done with this version of the theory, including a somewhat inconclusive effort to identify a tachyon vacuum [63, 126]. It is the full quantum closed string field theory that requires stubs. This is made clear by considering $\Delta\mathcal{V}_{0,3}$, where two coordinate curves of the vertex are twist glued. Without stubs, this produces singular surfaces [127]. With length π stubs, all nontrivial closed curves remain longer than 2π , and one can determine the quantum vertex $\mathcal{V}_{1,1}$ needed to satisfy the master equation.

我们注意到，只要我们仅研究经典闭弦场论，就不需要短柱——此时坐标曲线与多面体的面重合。该版本的理论已经有了相当多的研究工作，其中包括寻找快子真空的相关尝试，但至今没有定论 [63, 126]。完整量子闭弦场论才需要短柱。考察 $\Delta\mathcal{V}_{0,3}$ 就能明确这一点: 在 $\Delta\mathcal{V}_{0,3}$ 中，顶点的两条坐标曲线是扭粘合的。没有短柱时，这种粘合会产生奇异曲面 [127]。配备长度为 π 的短柱后，所有非平凡闭曲线的长度都保持大于 2π ，我们就能确定满足主方程所需的量子顶点 $\mathcal{V}_{1,1}$ 。

While the minimal area metrics clearly exist for all genus zero surfaces and across large parts of the moduli space of non-zero genus surfaces, there is still no mathematical proof that they exist in all cases. Perhaps existence is just difficult to prove, and the result will be established at some point. All minimal area metrics that arise from quadratic differentials are flat, except for negative curvature singularities at the zeroes of the differential. Moreover, on any region of the surface, there is just one band of geodesics saturating the length condition. For general surfaces where the minimal area metric does not arise from a quadratic differential, it has also not been clear if bulk curvature can exist or lines of curvature can exist. Moreover, it was expected that there would be regions of the surface foliated by multiple bands of geodesics.

虽然所有零亏格曲面、以及非零亏格模空间的大部分区域都明确存在极小面积度量，但目前还没有数学证明表明所有情况下极小面积度量都存在。或许存在性只是难以证明，未来总会得到结果。所有由二次微分产生的极小面积度量都是平坦的，仅在微分零点处存在负曲率奇点。此外，在曲面的任意区域上，都只有一簇测地线满足饱和长度条件。对于极小面积度量不由二次微分产生的一般曲面，目前仍不清楚是否可以存在体曲率或曲率线。此外，人们此前认为这类曲面会存在由多簇测地线叶化的区域。

Progress on the minimal area problem was made by M. Headrick and Zwiebach [128, 129], using the method of convex optimization. The original minimal-area problem was analyzed in terms of homology (as opposed to homotopy), using calibrations and the max flow-min cut theorem to formulate it as a local convex program that is easily implemented numerically. Moreover, a dual program, involving maximization of a concave functional was derived, providing lower bounds for the minimal area. These methods were applied to find (numerically) the minimal area metric for the once-punctured square torus. The solution displays regions covered by intersecting bands of length-saturating geodesics and exhibits both positive and negative bulk curvature. Further work to understand regions with multiple intersecting bands of geodesics was reported by Usman and Zwiebach [130]. It showed that a claim that regions with more than four such bands are flat is simply not true. All in all, these works provide evidence that minimal area metrics exist. The minimal area problem fits in the area of systolic geometry, where a number of important results have been established [131].

M. 赫德里克和茨维巴赫通过凸优化方法，在极小面积问题上取得了进展 [128, 129]。他们利用校准和最大流-最小割定理，从同调 (区别于同伦) 角度分析了原始极小面积问题，将其构造为易于数值实现的局部凸规划。此外，他们还推导出了一个涉及凹泛函最大化的对偶规划，为极小面积给出了下界。这些方法被用于 (数值) 求解一次穿孔正方形环面的极小面积度量。解显示出区域被饱和长度测地线的相交带覆盖，且体曲率同时存在正负号。乌斯曼和茨维巴赫在后续工作中报道了对多相交测地带区域的研究 [130]，他们指出“包含超过四条此类带的区域是平坦的”这一论断完全不成立。总而言之，这些工作为极小面积度量的存在提供了证据。极小面积问题属于 systolic 几何领域，该领域已得到许多重要结果 [131]。

The minimal area problem can also be used for open string field theory and for open-closed string field theory. If we consider diagrams with only external open strings, we are looking at Riemann surfaces of all genus, with $b \geq 1$ boundaries and a number m of boundary punctures. One can show that the Witten classical open string field theory produces correctly all the moduli spaces because such diagrams are solutions of a minimal area problem: the surfaces have the minimal area under the condition that any open curve of nontrivial homotopy (and boundary endpoints) have length bigger or equal to π [68]. For open-closed string field theory, we need to consider all genus surfaces with arbitrary numbers of open and closed string punctures. Here, a natural minimal area problem requires nontrivial open curves (with boundary endpoints) to be longer or equal to π , while nontrivial closed curves are longer or equal to 2π . In writing out this theory, one must add length π stubs to all vertices, including those of the open string, and for the boundary state vertex, which represents the one-point function of closed strings in a disk. Thus, the classical open string field theory subsector of the open-closed quantum theory is non-polynomial.

极小面积问题也可应用于开弦场论和开-闭弦场论。如果我们考虑仅带有外开弦的图，我们研究的是任意亏格、带有 $b \geq 1$ 个边界和 m 个边界穿孔的黎曼曲面。可以证明，威腾经典开弦场论能正确给出所有模空间，因为这类图正是极小面积问题的解: 在“任何非平凡同伦开曲线 (端点在边界上) 长度大于等于 π ”的条件下，这类曲面的面积最小 [68]。对于开-闭弦场论，我们需要考虑带有任意数量开弦穿孔和闭弦穿孔的任意亏格曲面。对此，一个自然的极小面积问题要求非平凡开曲线 (端点在边界上) 长度大于等于 π ，非平凡闭曲线长度大于等于 2π 。在构建该理论时，必须给所有顶点加上长度为 π 的短柱，包括开弦顶点，以及代表圆盘上闭弦单点函数的边界态顶点。因此，开-闭量子理论中的经典开弦场场子域是非多项式的。

String Vertices from Hyperbolic Metrics

双曲度量构造的弦顶点

Much is known about hyperbolic metrics on two-dimensional surfaces, metrics of Gaussian curvature -1 . The moduli space of such metrics on an orientable surface of genus g and n punctures coincides with the moduli spaces of conformal structures on the surface, namely, with the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces-this is a version of the uniformization theorem [132]. It is also interesting to note that surfaces that must be included in the definition of $\mathcal{V}$, as written out in Eq. (434), are precisely those for which hyperbolic metrics exist. Indeed, hyperbolic metrics do not exist for spheres with less than three punctures and for tori without punctures. These facts motivated Pius and Moosavian to explore the construction of closed string field theory using hyperbolic metrics on Riemann surfaces [133, 134]. They considered surfaces with punctures, and that led to a complication. Such metrics have cusps on the punctures. Moreover, the cutting and gluing of such metrics is not done along geodesics, and therefore it does not give a hyperbolic metric. These complications mean that the solution of the master equation is only approximate and must be improved.

人们对二维曲面的双曲度量 (即高斯曲率为-1 的度量) 已有充分了解。亏格为 g 且带有 n 个孔的可定向曲面上，这类度量的模空间与该曲面上共形结构的模空间一致，也就是黎曼曲面的模空间 $\mathcal{M}_{g,n}$ ——这是一致化定理的一种形式 [132]。同时值得注意的是，式 (434) 给出的 $\mathcal{V}$ 定义中需要包含的曲面，恰好是存在双曲度量的曲面。事实上，少于三个孔的球面和不带孔的环面都不存在双曲度量。这些事实推动 Pius 和 Moosavian 探索利用黎曼曲面上的双曲度量构造闭弦场论 [133, 134]。他们研究带孔曲面时遇到了一个难题: 这类度量在孔处存在尖点，而且这类度量的切割与粘合不是沿测地线进行的，因此无法得到双曲度量。这些难题意味着主方程的解只是近似解，仍需要优化。

In a somewhat different approach, Costello and Zwiebach [34] worked with surfaces of genus g and n boundary components, which allow for exact solutions of the master equation, as we will explain. On such surfaces, we consider hyperbolic metrics in which the n boundaries are geodesics, all of length L . Given any such surface $\sum_{g,n}$ with its metric, there is an obvious grafting map gr_∞ that takes it to $\widehat{\mathcal{P}}_{g,n}$, that is, a map to the space of Riemann surfaces of genus g with n punctures and local coordinates at the punctures. The map

Costello 和 Zwiebach[34] 采用了一种不同的思路，研究亏格 g 带 n 个边界分支的曲面，这类曲面可以得到主方程的精确解，我们稍后会对此说明。在这类曲面上，我们研究双曲度量，其中 n 个边界都是测地线，且长度均为 L 。对任意一个这类带度量的曲面 $\sum_{g,n}$ ，存在一个自然的嫁接映射 gr_∞ ，将它映射到 $\widehat{\mathcal{P}}_{g,n}$ ，也就是映射到亏格 g 带 n 个孔且孔处带有局部坐标的黎曼曲面空间。这个映射

$$\mathrm{gr}_\infty : \sum_{g,n} \to \widehat{\mathcal{P}}_{g,n} \tag{474}$$

simply attaches semi-infinite cylinders of circumference L to each of the boundaries of the surface. The semi-infinite cylinders define the coordinate disks for the punctures in the resulting surface, the two related by the exponential map.

只需将周长为 L 的半无限圆柱连接到曲面的每个边界上。半无限圆柱确定了结果曲面中孔的坐标圆盘，二者通过指数映射关联。

Let us call $\mathcal{M}_{g,n,L}$ the moduli space of hyperbolic metrics on a surface of genus g with n geodesic boundaries, all of length L . The grafting map, followed by the map π forgetting coordinates, is then a map

我们将亏格 g 带 n 个长度均为 L 的测地边界的曲面上双曲度量的模空间记为 $\mathcal{M}_{g,n,L}$ 。嫁接映射后接遗忘坐标的映射 π ，就得到了映射

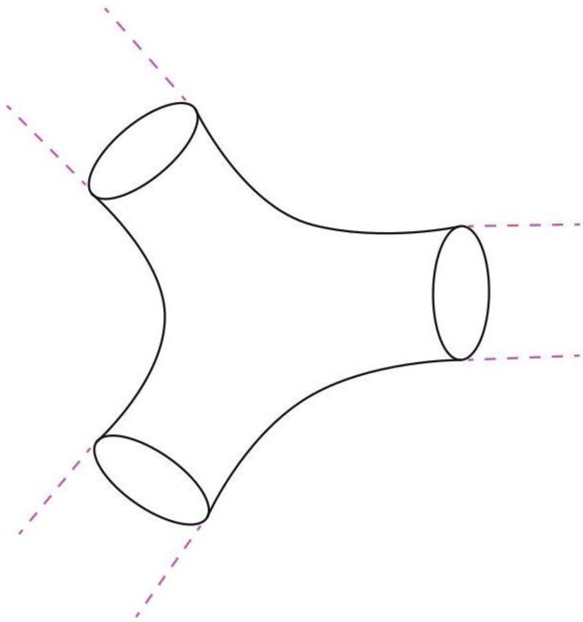
$$\pi \circ \mathrm{gr}_\infty : \mathcal{M}_{g,n,L} \to \mathcal{M}_{g,n} \tag{475}$$

It is a theorem that this map is in fact a homomorphism, a one-to-one onto map. By considering all the hyperbolic metrics on a genus g surface with n geodesic boundary components, we have created a cover of moduli space!

已有定理证明这个映射实际上是同胚，即一一满映射。通过研究亏格 g 带 n 个测地边界分支的曲面上所有的双曲度量，我们得到了模空间的一个覆盖！

Fig. 15 A sketch of a hyperbolic three-closed string vertex. The boundaries are geodesics of equal lengths

图 15 双曲三闭弦顶点示意图。边界是长度相等的测地线



The simplest example of this is $\mathcal{M}_{0,3,L}$, the moduli space of hyperbolic metrics on a sphere with three holes. The hyperbolic metric here is unique, so this moduli space is just a point. This is the familiar pants diagram shown in Fig. 15, with the dotted lines representing the flat cylinders that could be attached to the boundaries to create the surface $\mathcal{V}_{0,3} \in \widehat{\mathcal{P}}_{0,3}$. As we will see below, this is the closed string vertex for a hyperbolic string field theory.

最简单的例子是 $\mathcal{M}_{0,3,L}$ ，即带三个孔的球面上双曲度量的模空间。这里的双曲度量是唯一的，因此这个模空间只是一个点。这就是图 15 所示的熟知的裤状图，虚线代表可以连接到边界的平坦圆柱，用来构造曲面 $\mathcal{V}_{0,3} \in \widehat{\mathcal{P}}_{0,3}$ 。我们下文会看到，这就是双曲弦场论的闭弦顶点。

To proceed, we need one more definition. For a surface $\sum$ with boundaries and equipped with a metric, the systole of $\sum$, called sys $\sum$, is the length of the shortest non-contractible closed geodesic, which is not a boundary component. If we have a surface with no geodesics, except for the boundary geodesics of length L , the systole is declared to be L . In general, for a surface in $\mathcal{M}_{g,n,L}$, the systole can be larger than L , equal to L , if the shortest curve, which is not a boundary, has length exactly equal to the boundary, or shorter than L .

接下来我们需要再给出一个定义。对于带边界且装备了度量的曲面 $\sum$ ，$\sum$ 的收缩 syst $\sum$ 是指最短的不可缩非边界闭测地线的长度。如果曲面除长度为 L 的边界测地线外不存在其他测地线，则规定收缩为 L 。一般来说，对于 $\mathcal{M}_{g,n,L}$ 中的曲面，收缩可以大于 L 、等于 L (即最短非边界曲线的长度恰好等于边界长度) 或小于 L 。

On $\mathcal{M}_{g,n,L}$, we can now select a subset $\widetilde{\mathcal{V}}_{g,n}(L)$ that includes those surfaces where the systole is larger than or equal to L :

在 $\mathcal{M}_{g,n,L}$ 上，我们现在可以选出一个包含收缩 systole 大于等于 L 的曲面的子集 $\widetilde{\mathcal{V}}_{g,n}(L)$:

$$\widetilde{\mathcal{V}}_{g,n}(L) \equiv \left\{\sum \in \mathcal{M}_{g,n,L} \mid \text{ sys } \sum \geq L\right\}. \tag{476}$$

Since the boundary geodesics are of length L , the above sets include all surfaces that have no geodesic of length less than L . Even if there are no boundaries $(n = 0)$, the definition is sensible: here $\mathcal{M}_{g,0,L} = \mathcal{M}_{g,0}$, and the above defines a certain subset of $\mathcal{M}_{g,0}$. For $\mathcal{M}_{0,3,L}$, the selected subset is a single surface, the pants diagram with boundaries of length L .

由于边界测地线的长度为 L ，上述集合包含了所有不存在长度小于 L 的测地线的曲面。即使没有边界 $(n = 0)$ ，该定义仍然合理: 此时为 $\mathcal{M}_{g,0,L} = \mathcal{M}_{g,0}$ ，上述定义给出了 $\mathcal{M}_{g,0}$ 的一个确定子集。当 $\mathcal{M}_{0,3,L}$ 时，选出的子集仅包含一个曲面，即边界长度为 L 的裤形图。

The string vertices $\mathcal{V}_{g,n}(L)$ are now defined as sets $\widetilde{\mathcal{V}}_{g,n}(L)$ grafted with semi-infinite cylinders to turn the boundaries into punctures with local coordinates:

弦顶点 $\mathcal{V}_{g,n}(L)$ 现在被定义为拼接了半无限柱面的集合 $\widetilde{\mathcal{V}}_{g,n}(L)$ ，以此将边界变为带局部坐标的孔:

$$\mathcal{V}_{g,n}(L) \equiv \mathrm{gr}_{\infty}\left(\widetilde{\mathcal{V}}_{g,n}(L)\right). \tag{477}$$

The main result of [34] is now simple to state:

文献 [34] 的主要结论现在可以简洁表述为:

The sets $\mathcal{V}_{g,n}(L)$, with $L \leq L_* = 2\sinh^{-1}1$, solve the geometric master equation.

满足 $L \leq L_* = 2\sinh^{-1}1$ 的集合 $\mathcal{V}_{g,n}(L)$ 满足几何主方程。

This means we get a solution for boundary lengths that are bounded by the constant $L_* \simeq 1.76275$.

这意味着我们得到了边界长度被常数 $L_* \simeq 1.76275$ 限制的解。

We will not go over the details of the proof that establishes the above result, but the intuition for it comes from familiar facts about collars in hyperbolic metrics (see, e.g.,[132]). Consider a simple closed geodesic γ on a Riemann surface. The collar $\mathcal{C}(\gamma)$ of width w about γ is the set of all points on the surface whose distance to the curve γ does not exceed $w/2$. For a geodesic on the interior of the surface, the collar extends to the two sides of it. For a geodesic boundary, the collar just extends to one side of it, so that its "width" is in fact $w/2$. Given a collection of simple closed geodesics γ_i of length ℓ_i that do not intersect, we can consider collars of width w_i chosen such that

我们不会赘述证明上述结论的细节，但其直观理解来自双曲度量中领圈定理的常见结论 (例如参见 [132])。考虑黎曼曲面上的一条简单闭测地线 γ 。围绕 γ 、宽度为 w 的领圈 $\mathcal{C}(\gamma)$ 是曲面上所有到曲线 γ 的距离不超过 $w/2$ 的点的集合。对于曲面内部的测地线，领圈向测地线的两侧延伸。对于边界测地线，领圈仅向一侧延伸，因此其“宽度”实际为 $w/2$ 。给定一组长度为 ℓ_i 、互不相交的简单闭测地线 γ_i ，我们可以考虑宽度取为 w_i 的领圈，满足

$$\sinh\left(\frac{1}{2}w_i\right)\sinh\left(\frac{1}{2}\ell_i\right) = 1. \tag{478}$$

It is a remarkable result in hyperbolic geometry that such collars are disjoint on the surface; they do not intersect! The collars are annuli that exist around each geodesic. The shorter the geodesic, the wider the collar. The width w of the collar equals the length ℓ of the geodesic when both factors in the above left-hand side are equal to one, that is, for $\ell = L_* = 2\sinh^{-1} 1$. It follows that for a geodesic of length $L < L_*$, its collar will have $w_L > L_* > L$; thus:

双曲几何中有一个很出色的结论: 这类领圈在曲面上互不相交！每个测地线周围的领圈都是不相交的环形区域。测地线越短，对应的领圈越宽。当上述左手边的两个因子都等于 1，也就是对于 $\ell = L_* = 2\sinh^{-1} 1$ ，领圈的宽度 w 等于测地线的长度 ℓ 。由此可得，对于长度为 $L < L_*$ 的测地线，其领圈满足 $w_L > L_* > L$ ；因此:

$$L \leq L_* \ \rightarrow \ w \geq L \tag{479}$$

This is why the above subsets $\mathcal{V}_{g,n}(L)$ satisfy the master equation with $L \leq L_*$. In that case, the collars around the geodesic boundaries have width greater than $L/2$, so that when they are glued together, any non-trivial closed curve crossing the seam will have length greater than L , and thus the gluing will result on a surface on the vertex region. Furthermore, since the glued surface now has systole L , realized by the gluing curve, it lies on the boundary of the vertex region. As one can see, the collars here play the role of the stubs in the minimal area problem, whose role was to prevent the creation of short non-contractible closed curves upon gluing.

这就是上述子集 $\mathcal{V}_{g,n}(L)$ 满足带 $L \leq L_*$ 的主方程的原因。此时，边界测地线周围的领圈宽度大于 $L/2$ ，因此当它们粘合在一起时，任何穿过接缝的非平凡闭曲线长度都会大于 L ，粘合得到的曲面也就属于顶点区域。进一步，由于粘合后曲面的收缩 systole 恰好为 L (由粘合曲线实现)，该曲面位于顶点区域的边界上。不难看出，此处领圈的作用等价于极小面积问题中的短柱 stub，作用是防止粘合过程产生短的不可收缩闭曲线。

The above construction of the closed string vertices is, at this point, the only fully rigorous and explicit construction available. Given the wealth of results known about hyperbolic metrics, one would expect that in some way, hyperbolic string field theory could be tractable analytically.

目前上述闭弦顶点的构造是唯一可用的完全严格且显式的构造。鉴于关于双曲度量已有非常丰富的研究结果，可以预期双曲弦场理论在某种程度上是可以解析求解的。

This hyperbolic construction has been extended to open-closed string field theory by Cho [135]. The Riemann surfaces here have borders, geodesic closed curves of length L_c where we attach external closed string semi-infinite cylinders; open-puncture boundaries, geodesic segments of length L_o where we attach external open string semi-infinite strips; and just plain boundaries in the sense of open string theory. Moreover, where boundaries and open-puncture boundaries meet, they are orthogonal. The string vertices for any moduli space (with two exceptions noted below) are the surfaces for which the hyperbolic metric has closed curve systole exceeding or equal to L_c and open curve systole exceeding or equal to L_o . The closed curve systole was defined before. The open curve systole is the length of the shortest noncontractible open curve with endpoints on boundaries that is not itself an open-puncture boundary. For the vertices to satisfy the BV geometric master equation, the values of L_c and L_o must satisfy the conditions

赵 [135] 已将这种双曲构造推广到开-闭弦场论。此处的黎曼曲面具有以下边界: 我们外接外闭弦半无限柱面处，是长度为 L_c 的测地闭曲线；开刺边界处，是我们外接外开弦半无限带处，长度为 L_o 的测地线段；以及开弦理论意义下的普通边界。此外，普通边界与开刺边界相交处二者正交。任意模空间的弦顶点 (下述两种例外情况除外) 均满足: 双曲度量的闭曲线收缩压大于等于 L_c，开曲线收缩压大于等于 L_o。闭曲线收缩压已有前述定义。开曲线收缩压指端点在边界上、本身不是开刺边界的最短不可缩开曲线的长度。为让顶点满足 BV 几何主方程，L_c 和 L_o 的值必须满足条件

$$L_c \leq L'_*, \text{ and } \sinh L_c \sinh L_o \leq 1. \tag{480}$$

The constant L'_* is defined by the condition $\sinh(L'_*/2)\sinh L'_* = 1$ giving $L'_* \simeq 1.21876$. One can see that for $L_o \leq L'_*/2$, all $L_c \leq L'_*$ work. For $L_o > L'_*/2$, the possible values of L_c have a smaller upper bound. The above rule for vertices does not apply for two moduli spaces: for the disk with one bulk puncture, the vertex is just a circle of length L_c (the limit of a cylinder of circumference L_c and vanishing height, with a border and a boundary), and for the annulus without punctures, there is no vertex.

常数 L'_* 由条件 $\sinh(L'_*/2)\sinh L'_* = 1$ 定义，由此可得 $L'_* \simeq$ 为 1.21876。不难看出，对 $L_o \leq L'_*/2$ 而言，所有 $L_c \leq L'_*$ 都成立。对 $L_o > L'_*/2$ 而言，L_c 的可能取值拥有更小的上界。上述顶点规则不适用于两种模空间: 对带一个整体刺的圆盘而言，其顶点就是长度为 L_c 的圆 (周长为 L_c、高度趋于零的圆柱，带有一个外边界和一个内边界的极限)；对不带刺的环面而言，不存在顶点。

The explicit calculation of the local coordinates of the hyperbolic three-string vertex has been done by Firat [136]. This allows for off-shell computations with this vertex. In fact, the result gives the coordinates for a general hyperbolic pants diagram, that is, a sphere with three geodesic boundaries of different lengths. This is useful because the Teichmuller moduli space of hyperbolic surfaces has a nice decomposition in terms of the gluing of pants. The Liouville equation for the conformal factor of the metric on the three-holed sphere is solved for by relating it to the monodromy problem for a second-order linear ordinary differential equation, a Fuchsian equation on the complex plane.

菲拉特 [136] 已完成双曲三弦顶点局部坐标的显式计算，这使得可以用该顶点进行离壳计算。实际上，该结果给出了一般双曲裤图的坐标，也就是带有三个不同长度测地边界的球面。这很有用，因为双曲曲面的泰希米勒模空间可以很好地分解为裤形粘合。通过将三孔球面度量共形因子的刘维尔方程关联到复平面上二阶线性常微分方程 (富克斯方程) 的单值问题，我们已求解了该方程。

Hyperbolic vertices for closed string field theory have an interesting limit. As long as we focus on the genus zero theory, the condition $L < L_*$ is not really required, and L can be taken as large as one wishes. In fact, when $L \to \infty$, the hyperbolic vertices approach, up to an overall constant scale factor, the polyhedral vertices of classical string field theory [34]. This has been exploited in [123], showing that indeed, the Strebel quadratic differentials associated with the polyhedra arise from hyperbolic vertices as a WKB approximation applied to the Fuchsian equation. This allows for the computation of these quadratic differentials using the tools of Liouville conformal field theory, in particular, their conformal blocks.

闭弦场论的双曲顶点存在一个有意思的极限。只要我们聚焦于零亏格理论，条件 $L < L_*$ 并非必需，且 L 可以取任意大的值。实际上，当 $L \to \infty$ 时，除整体常数标度因子外，双曲顶点趋近于经典弦场论的多面体顶点 [34]。文献 [123] 利用这一点证明，与多面体关联的施特雷贝尔二次微分确实是通过对富克斯方程应用 WKB 近似，从双曲顶点得到的。这使得我们可以利用刘维尔共形场论的工具 (尤其是共形块) 计算这些二次微分。

Finally, for useful information derived about the properties of off-shell amplitudes in hyperbolic geometry, see [133, 134].

最后，关于从双曲几何中离壳振幅性质得到的有用结论，参见 [133, 134]。

String Vertices for Open Superstring Field Theory

开超弦场论的弦顶点

If we restrict ourselves to the NS sector of the classical open superstring field theory, the original proposal of [137] was to use a string field in the minus one picture, with the standard BRST operator for the kinetic term. The cubic interaction requires a picture changing operator adding one unit of picture number so that the total picture number in the correlator is equal to minus two, as required. Using the associative vertex, the PCO was inserted at the string midpoint, but this leads to difficulties: associativity fails to hold as one gets a singular collision of two picture changing operators. While several ways to address this difficulty have been considered, a canonical way to insert picture changing operators that does not suffer from singularities was developed by Erler, Konopka, and Sachs in [28, 59, 60] . The result is a consistent theory for the NS sector of classical open superstrings, the NS sector of classical heterotic strings, and the NS-NS sector of classical type II superstrings. Our focus here will be on the NS sector of open superstrings discussed in [59]. We will comment on the extension for heterotic strings and for type II superstrings.

如果我们将讨论范围限定在经典开超弦场论的 NS 扇区，文献 [137] 的原始方案是: 动能项采用负 1 图的弦场和标准 BRST 算符。三次相互作用需要一个图变算符来增加 1 个单位的图数，从而使关联函数中的总图数满足要求，等于负二。在结合顶点中，PCO 被插入在弦中点，但这会引发问题: 当两个图变算符发生奇异碰撞时，结合性不再成立。虽然人们已经研究了多种解决该问题的方法，Erler、Konopka 和 Sachs 在 [28, 59, 60] 中提出了一种不会产生奇异性的插入图变算符的正则方法。由此得到了适用于经典开超弦 NS 扇区、经典杂化弦 NS 扇区以及经典 II 型超弦 NS-NS 扇区的自洽理论。本文我们将重点讨论文献 [59] 中研究的开超弦 NS 扇区，最后会对杂化弦和 II 型超弦的推广做简要评述。

For open superstring field theory, the idea is to begin with a series of bosonic open string products $b^{(0)}_{n+1}$, with $n \geq 0$, satisfying the A_∞ axioms. The superscript zero in the products indicates no insertion of picture changing operators; the subscript is the number of states going into the product. An explicit procedure using canonical insertions constructs a new series of open superstring products $b^{(n)}_{n+1}$. Here, the superscript indicates the extra picture number n carried by such insertions, which is appropriate for open superstring interactions of $n+1$NS fields, each of picture minus one. These products, by construction, still define an A_∞ algebra, and thus the open superstring action can be written as usual. For convenience of notation, we use the ”bold” operators $\mathbf{b}^{(0)}_{n+1}$ and $\mathbf{b}^{(n)}_{n+1}$ to denote the extension of the products to act on the tensor co-algebra $T(\mathcal{H}_o)$ (see section ” A_∞ Algebras and Classical Open String Field Theory”). We define the formal sum of superstring products, an operator of degree one,

对于开超弦场论，其基本思路是从满足 A_∞ 公理的一系列玻色开弦乘积 $b_{n+1}^{(0)}$ (满足 $n \geq 0$) 出发。乘积的上标零表示不插入图变算子，下标表示参与乘积的态数目。利用标准插入的显式构造，可以得到一组新的开超弦乘积序列 $b_{n+1}^{(n)}$ 。此处上标代表这类插入携带的额外图数 n ，这对于 $n+1$NS 场的开超弦相互作用是适用的，其中每个场的图数都是负一。通过构造，这些乘积仍然定义了一个 A_∞ 代数，因此开超弦作用量可以按常规形式写出。为了简化记号，我们用“粗体”算符 $\mathbf{b}_{n+1}^{(0)}$ 和 $\mathbf{b}_{n+1}^{(n)}$ 表示乘积推广后作用在张量余代数 $T(\mathcal{H}_o)$ 上的结果 (参见章节“A_∞ 代数与经典开弦场理论”)。我们将超弦乘积的形式和定义为一次齐次算符,

$$\mathbf{B}^{[0]}(t) = \sum_{n=0}^{\infty} t^n \mathbf{b}_{n+1}^{(n)}. \tag{481}$$

The use of the formal parameter t as opposed to just doing the sum of the products (as we did for open bosonic strings) will be convenient for the later discussion. The superscript [0] indicates that the number of picture changing operators is the correct one for the products, assuming an NS field of picture minus one. Thus the ”zero” deficit is denoted by the superscript. These vertices are consistent if they satisfy the A_∞ condition:

使用形式参数 t 而非直接对乘积求和 (就像我们对开玻色弦所做的那样) 对后续讨论会更方便。上标 [0] 表示，对于负一图的 NS 场，乘积所需的图变算子数目恰好正确，因此上标标记了“零”亏缺。若这些顶点满足 A_∞ 条件，就是自洽的:

$$\left(\mathbf{B}^{[0]}(t)\right)^2 = 0. \tag{482}$$

Additionally, we require that these vertices belong to the ”small” Hilbert space: acting on string fields killed by η_0 . Defining the associated η_0 operator acting on the tensor algebra, this condition takes the form

此外，我们要求这些顶点属于“小”希尔伯特空间: 即作用在被 η_0 零化的弦场上。将作用在张量代数上的对应 η_0 算符定义后，该条件可以写为

$$\left\{\boldsymbol{\eta}_0, \mathbf{B}^{[0]}(t)\right\} = 0. \tag{483}$$

This just states that η_0 should be a derivation of the superstring products. Of course, η_0 is also a derivation of the bosonic string products; this means that $\left\{\boldsymbol{\eta}_0, \mathbf{b}_n^{(0)}\right\} = 0$, for all $n \geq 1$. In order to solve for the $\mathbf{b}_{n+1}^{(n)}$ operators satisfying these conditions, one introduces a larger set of products of various deficits, so that one can incorporate into a single framework the bosonic products of maximal deficit. For string products with picture number deficit m , we have

这仅说明 η_0 应当是超弦乘积的导子。当然，η_0 也是玻色弦乘积的导子，这意味着对所有 $n \geq 1$ 都有 $\left\{\boldsymbol{\eta}_0, \mathbf{b}_n^{(0)}\right\} = 0$ 成立。为了求解满足这些条件的 $\mathbf{b}_{n+1}^{(n)}$ 算符，我们引入更大一类带有不同亏缺的乘积，从而可以将最大亏缺的玻色乘积纳入同一个框架。对于图数亏缺为 m 的弦乘积，我们有

$$\mathbf{B}^{[m]}(t) = \sum_{n=0}^{\infty} t^n \mathbf{b}^{(n)}_{m+n+1} m = 0, 1, \ldots \tag{484}$$

In all generality, we now put together all such products

现在我们将所有这类乘积整合起来，得到最一般的形式

$$\mathbf{B}(s,t) = \sum_{m=0}^{\infty} s^m \mathbf{B}^{[m]}(t) = \sum_{m,n=0}^{\infty} s^m t^n \mathbf{b}^{(n)}_{m+n+1}. \tag{485}$$

Note that $\mathbf{B}(0,t)$ is the superstring collection of products, while $\mathbf{B}(s,0) = \sum_m s^m \mathbf{b}^{(0)}_{m+1}$ is the bosonic string collection of products. Moreover, one introduces a set of Grassmann even operators $\boldsymbol{\mu}$ that change the picture numbers. We include variable number, greater or equal to two, of input string states and a variable number, greater or equal to one, of picture number insertions:

注意 $\mathbf{B}(0,t)$ 是超弦的乘积集合，而 $\mathbf{B}(s,0) = \sum_m s^m \mathbf{b}^{(0)}_{m+1}$ 是玻色弦的乘积集合。此外，我们引入一组格拉斯曼偶算符 $\boldsymbol{\mu}$ 来改变图数。我们包含数量可变、不小于 2 的输入弦态，以及数量可变、不小于 1 的图数插入:

$$\boldsymbol{\mu}(s,t) = \sum_{m=0}^{\infty} s^m \boldsymbol{\mu}^{[m]}(t) = \sum_{m,n=0}^{\infty} s^m t^n \boldsymbol{\mu}^{(n+1)}_{m+n+2}. \tag{486}$$

At this point, what is needed is guaranteed if the following equations hold:

此时，只要满足以下方程，就能得到我们需要的结论:

$$\{\mathbf{B}(s,t), \mathbf{B}(s,t)\} = 0, \text{ and } \{\boldsymbol{\eta}_0, \mathbf{B}(s,t)\} = 0. \tag{487}$$

For $t = 0$, these manifestly hold, because they are the constraints satisfied by the bosonic string products. For $s = 0$, these are the relations we need to hold for the superstring products. The insight of [59] is the proposal of a set of recursion relations expressed as a set of differential equations:

对于 $t = 0$ ，这些方程显然成立，因为它们正是玻色弦乘积满足的约束。对于 $s = 0$ ，这些就是我们要求超弦乘积满足的关系。文献 [59] 的洞见在于提出了一组表示为微分方程的递推关系:

$$\frac{\partial}{\partial t}\mathbf{B}(s,t) = [\mathbf{B}(s,t), \boldsymbol{\mu}(s,t)], \ \frac{\partial}{\partial s}\mathbf{B}(s,t) = [\boldsymbol{\eta}_0, \boldsymbol{\mu}(s,t)]. \tag{488}$$

These equations allow for a recursive solution that yields, at the end, the superstring vertices. But, more crucially, they guarantee that the required Eqs. (487) hold. Indeed, this follows by taking the t derivatives of the left-hand sides of (487), and showing that given that these left-hand sides vanish for $t = 0$, they must vanish for all t . The algorithm to find a solution to these equations was described in [59].

这些方程允许递推求解，最终得到超弦顶点。但更关键的是，它们保证了所需的式 (487) 成立。事实上，这可以通过对式 (487) 的左侧取 t 导数来证明: 若这些左侧在 $t = 0$ 时为零，则它们对所有 t 都必然为零。求解这些方程的算法已在文献 [59] 中给出。

We illustrate the recursive procedure with the lowest order case, the one fixing the first product $\mathbf{b}_2^{(1)}$ of the open superstring. Recalling that $\mathbf{b}_1^{(0)} = \mathbf{Q}$, we write

我们用最低阶的情况举例说明这个递推过程，即确定开超弦第一个乘积 $\mathbf{b}_2^{(1)}$ 的情况。回顾 $\mathbf{b}_1^{(0)} = \mathbf{Q}$ ，我们写出

$$\mathbf{B}(s,t) = \mathbf{Q} + t\mathbf{b}_2^{(1)} + s\mathbf{b}_2^{(0)} + \ldots, \ \boldsymbol{\mu}(s,t) = \boldsymbol{\mu}_2^{(1)} + \ldots \tag{489}$$

The two differential equations above give, to leading order, two equations

上述两个微分方程在领头阶给出两个方程

$$\mathbf{b}_2^{(1)} = \left[\mathbf{Q}, \boldsymbol{\mu}_2^{(1)}\right], \ \mathbf{b}_2^{(0)} = \left[\boldsymbol{\eta}_0, \boldsymbol{\mu}_2^{(1)}\right]. \tag{490}$$

With a little trial and error, the solution for $\boldsymbol{\mu}_2^{(1)}$ can be written in terms of $\boldsymbol{\xi}_0$ and the product $\mathbf{b}_2^{(0)}$:

经过少量试错，$\boldsymbol{\mu}_2^{(1)}$ 的解可以用 $\boldsymbol{\xi}_0$ 和乘积 $\mathbf{b}_2^{(0)}$ 写出:

$$\boldsymbol{\mu}_2^{(1)} = \frac{1}{3}\left(\boldsymbol{\xi}_0\mathbf{b}_2^{(0)} - \mathbf{b}_2^{(0)}\boldsymbol{\xi}_0\right). \tag{491}$$

As a two-product, this is

作为两乘积，它写为

$$\boldsymbol{\mu}_2^{(1)}(A \otimes B) = \frac{1}{3}\left(\xi_0(A,B) - (\xi_0 A, B) - (-1)^A (A, \xi_0 B)\right). \tag{492}$$

This solution for $\boldsymbol{\mu}_2^{(1)}$ is easily confirmed acting on two string fields A, B in the small Hilbert space:

将 $\boldsymbol{\mu}_2^{(1)}$ 的这个解作用在小希尔伯特空间中的两个弦场 A, B 上，很容易验证其正确性:

$$\left[\boldsymbol{\eta}_0, \boldsymbol{\mu}_2^{(1)}\right](A \otimes B) = \boldsymbol{\eta}_0\boldsymbol{\mu}_2^{(1)}(A \otimes B)$$

$$= \frac{1}{3}\eta_0\left(\xi_0(A,B) - (\xi_0 A, B) - (-1)^A (A, \xi_0 B)\right)$$

$$= \frac{1}{3}\left(\eta_0\xi_0(A,B) + (\eta_0\xi_0 A, B) + (A, \eta_0\xi_0 B)\right)$$

$$= \frac{1}{3}\left((A,B) + (A,B) + (A,B)\right) = (A,B) = \mathbf{b}_2^{(0)}(A \otimes B).$$

(493)

Finally, we can determine the superstring product

最后，我们可以确定超弦乘积

$$(A,B)^* \equiv \mathbf{b}_2^{(1)}(A\otimes B) = \left[\mathbf{Q},\boldsymbol{\mu}_2^{(1)}\right](A\otimes B)$$

$$= \mathbf{Q}\boldsymbol{\mu}_2^{(1)}(A\otimes B) - \boldsymbol{\mu}_2^{(1)}(QA\otimes B) - (-1)^A\boldsymbol{\mu}_2^{(1)}(A\otimes QB)$$

$$= \frac{1}{3}Q\left(\xi_0(A,B) - (\xi_0 A,B) - (-1)^A(A,\xi_0 B)\right) \tag{494}$$

$$-\frac{1}{3}\left(\xi_0(QA,B) - (\xi_0 QA,B) + (-1)^A(QA,\xi_0 B)\right)$$

$$-\frac{1}{3}(-1)^A\left(\xi_0(A,QB) - (\xi_0 A,QB) - (-1)^A(A,\xi_0 QB)\right).$$

Using the derivation property of Q and $\{Q,\xi_0\} = \mathcal{X}_0$, we quickly get

利用 Q 和 $\{Q,\xi_0\} = \mathcal{X}_0$ 的导数性质，我们很快得到

$$(A,B)^* \equiv \frac{1}{3}\left(\mathcal{X}_0(A,B) + (\mathcal{X}_0 A,B) + (A,\mathcal{X}_0 B)\right). \tag{495}$$

The calculation of the next product $(A,B,C)^*$ follows a similar line, although it is computationally harder. The construction, however, guarantees that all the higher superstring products exist, are in the small Hilbert space, and satisfy the A_∞ relations. Since the construction guarantees that we get satisfactory vertices, the action takes the expected form. With an open NS string field Φ of ghost number one and picture number minus one, and in the small Hilbert space, the action is

下一个乘积 $(A,B,C)^*$ 的计算遵循类似的思路，只是计算难度更大。但该构造保证所有高阶超弦乘积都存在，属于小希尔伯特空间，并且满足 A_∞ 关系。由于该构造保证我们能得到合格的顶点，作用量也就取预期的形式。对于鬼数为 1、图数为-1、属于小希尔伯特空间的开 NS 弦场 Φ，作用量为

$$S(\Phi) = \frac{1}{2}\langle\Phi,Q\Phi\rangle' + \sum_{n=1}^{\infty}\frac{1}{n+2}\langle\Phi,b_{n+1}^{(n)}(\Phi,\cdots,\Phi)\rangle'. \tag{496}$$

It should be noted that for open strings, the use of $\boldsymbol{\xi}_0$ is not mandatory. It suffices to use an operator $\boldsymbol{\xi}$ whose anticommutator with $\boldsymbol{\eta}_0$ is equal to one.

需要注意的是，对于开弦，并非必须使用 $\boldsymbol{\xi}_0$。只需使用一个满足与 $\boldsymbol{\eta}_0$ 的反对易子等于 1 的算符 $\boldsymbol{\xi}$ 就足够了。

For heterotic string field theory, an almost identical procedure works. The bosonic closed string products form an L_∞ algebra, and the heterotic closed string products, constructed with insertions on the bosonic products, also satisfy the L_∞ algebra while still living in the small Hilbert space. This time, it is important to use $\boldsymbol{\xi}_0$, since the resulting $\mathcal{X}_0$ insertions commute with the operators b_0^- and L_0^- . As a result, the new products will still satisfy the required constraints $b_0^- = 0$ and $L_0^- = 0$. The expression for the second product is completely analogous to the one found above for open superstrings:

对于杂化弦场论，几乎完全相同的过程也适用。玻色闭弦乘积构成一个 L_∞ 代数，在玻色乘积基础上通过插入构造得到的杂化闭弦乘积，同样满足 L_∞ 代数，并且仍属于小希尔伯特空间。这一次，使用 ξ_0 很重要，因为得到的 $\mathcal{X}_0$ 插入与算符 b_0^- 和 L_0^- 对易。因此，新乘积仍然满足所需的约束 $b_0^- = 0$ 和 $L_0^- = 0$ 。二阶乘积的表达式与上文开超弦得到的结果完全类似:

$$[A,B]^* \equiv \frac{1}{3}\left(\mathcal{X}_0\,[A,B] + [\mathcal{X}_0 A,B] + [A,\mathcal{X}_0 B]\right). \tag{497}$$

Here $[A,B]$ is the original bosonic string product. For type II superstrings, there are a couple of options in the way one treats left-moving and right-moving pictures. Starting with the bosonic products, an asymmetric approach first raises by recursion the left-moving picture number and then raises the right-moving picture number. A more symmetric approach treats both picture numbers as equivalent. For more details, see [28].

此处 $[A,B]$ 是原玻色弦乘积。对于 II 型超弦，处理左行和右行图的方式有几种可选方案。从玻色乘积出发，非对称方法先通过递归提高左行图数，再提高右行图数；更对称的方法则将两种图数同等处理。更多细节参见文献 [28]。

A relation between this approach and the vertical integration introduced in section "Bosonic String Amplitudes and Their Off-Shell Generalization" has been discussed in [138]. Inclusion of Ramond sector in this construction is possible but needs some extra ingredients. Reference [60] generalized the construction of open superstring field theory to describe equations of motion of the R-sector fields. While constructing an action along this line is also possible [139,140], this requires inclusion of an extra free field as in section "Type II Superstring Field Theory" or using a restricted vector space for Ramond sector fields as will be discussed in section "Open Superstring Field Theory in the Ramond Sector". Generalizations of such construction to the Ramond sector of heterotic and type II string theory have also been discussed in [141-143].

[138] 已经讨论了该方法与“玻色弦振幅及其离壳推广”一节引入的竖直积分之间的关联。该构造可以纳入拉蒙德 sector，但需要一些额外要素。文献 [60] 将开超弦场论的构造推广，用以描述 R sector 场的运动方程。虽然沿这一思路构造作用量也是可行的 [139,140]，但这需要像“II 型超弦场论”一节那样引入一个额外自由场，或是对拉蒙德 sector 场使用受限向量空间，这一点将在“拉蒙德 sector 的开超弦场论”一节讨论。该构造也已经在 [141-143] 中被推广至杂化弦与 II 型弦的拉蒙德 sector。

Some Subtleties in Quantum Closed Superstring Field Theory

量子闭超弦场论中的若干微妙问题

All constructions of $\mathcal{V}_{g,n}$ for bosonic string theory, as described in sections "Minimal Area String Vertices: Witten Vertex and Closed String Polyhedra" and "String Vertices from Hyperbolic Metrics", have the property that the local coordinates vary continuously as we move in the moduli space. In other words, $\mathcal{V}_{g,n}$ describes the piece of a submanifold of $\widehat{\mathcal{P}}_{g,n}$. This may not be possible for open and/or closed superstring field theories due to the existence of spurious poles. As reviewed in section "Bosonic String Amplitudes and Their Off-Shell Generalization", the only known systematic way of choosing the PCO locations as we move in the moduli space is to divide the moduli space into small chambers, choose PCO locations inside a given chamber avoiding spurious poles, and then add suitable compensating terms at the boundary between the two chambers. Therefore, the choice of $\mathcal{V}_{g,n}$ in string field theory should also reflect this. Given a choice of

$(6g-6+2n)$ - dimensional subspace $\mathcal{V}_{g,n}$ of $\widehat{\mathcal{P}}_{g,n}$ for the bosonic string field theory, we divide it into small chambers and, in each chamber, choose the PCO locations such that we avoid the spurious poles. In the $(6g-7+2n)$ -dimensional boundary between two chambers, we add vertical segments that interpolate between two sets of PCO locations on two sides of the boundary, but the vertical segments themselves must be a union of several segments so that along each of these segments, only one PCO location changes. As discussed in section "Bosonic String Amplitudes and Their Off-Shell Generalization", when such walls meet on a $(6g-8+2n)$ -dimensional subspace, we have to add new two-dimensional vertical segments. This process continues until we have ensured that the union of the original chambers and the vertical segments fill a ball-shaped region in $\widehat{\mathcal{P}}_{g,n}$ with no gaps. The rules for carrying out the integrals along the vertical segments to construct the products $\{A_1, \cdots, A_n\}$ are the same as for the construction of the amplitudes as discussed in section "Bosonic String Amplitudes and Their Off-Shell Generalization".

正如在“极小面积弦顶点: 威滕顶点与闭弦多面体”和“双曲度量导出的弦顶点”两节中所述，玻色弦理论中所有 $\mathcal{V}_{g,n}$ 的构造都具备局部坐标随模空间移动连续变化的性质。换句话说，$\mathcal{V}_{g,n}$ 描述了 $\widehat{\mathcal{P}}_{g,n}$ 的一个子流形片段。由于假极点的存在，这一点对于开弦和/或闭超弦场论未必成立。正如“玻色弦振幅及其离壳推广”一节中回顾的，目前已知在模空间移动时选择 PCO 位置的唯一系统方法，是将模空间划分为小腔室，在每个腔室内选择避开假极点的 PCO 位置，再在两个腔室的边界添加合适的补偿项。因此，弦场论中 $\mathcal{V}_{g,n}$ 的选择也应当反映这一点。给定玻色弦理论中 $\widehat{\mathcal{P}}_{g,n}$ 的一个 $(6g-6+2n)$ 维子空间 $\mathcal{V}_{g,n}$，我们将其划分为小腔室，并在每个腔室中选择避开假极点的 PCO 位置。在两个腔室之间的 $(6g-7+2n)$ 维边界上，我们添加垂线段来插值边界两侧的两组 PCO 位置，但垂线段本身必须是若干段的并集，使得每一段上只有一个 PCO 位置发生变化。正如“玻色弦振幅及其离壳推广”一节中讨论的，当此类壁相交于 $(6g-8+2n)$ 维子空间时，我们必须添加新的二维垂线段。这一过程持续进行，直到我们确保原始腔室与垂线段的并集填满 $\widehat{\mathcal{P}}_{g,n}$ 中一个无空隙的球形区域。沿垂线段积分构造乘积 $\{A_1, \cdots, A_n\}$ 的规则，与“玻色弦振幅及其离壳推广”一节中讨论的振幅构造规则完全相同。

There are however two more subtleties in this construction. First, we cannot construct $\mathcal{V}_{g,n}$ satisfying the geometric master equation (104) with a single choice of PCO locations in each chamber, since, as discussed above (124), the definition of Δ and $\{\ ,\ \} in\,(104)\,nownivovesinsertionof theoperator \mathcal{G}$, which, in the R sector, involves averages of PCO locations. Therefore, at least close to the boundaries, $\mathcal{V}_{g,n}$ ' s must also be chains that involve averages of subspaces of $\widehat{\mathcal{P}}_{g,n}$. This is also necessary for the interaction vertices $\mathcal{V}_{g,n}$ to be symmetric under the exchange of external punctures. For example, even for the three-point function on the sphere with NS sector external punctures, where we need one PCO for the heterotic theory and one PCO each in the holomorphic and anti-holomorphic sector in type II theories, it is not possible to find a single location that remains invariant under the $SL(2, C)$ transformations that permute the three punctures. So we must take the averages of different choices of PCO locations.

然而这一构造还存在另外两处微妙问题。首先，我们无法在每个腔室仅用单一 PCO 位置选择构造出满足几何主方程 (104) 的 $\mathcal{V}_{g,n}$，因为正如上文 (124) 所讨论的，Δ 和 $\{$ 、$\} in\,(104)\,nownivovesinsertionof theoperator \mathcal{G}$ 的定义在 Rsector 中涉及 PCO 位置的平均。因此，至少在边界附近，$\mathcal{V}_{g,n}$ 必须是包含 $\widehat{\mathcal{P}}_{g,n}$ 子空间平均的链。这对于相互作用顶点 $\mathcal{V}_{g,n}$ 在外部穿刺交换下保持对称性也是必要的。例如，即便对球面 NS sector 外部穿刺的三点函数而言——杂化理论中这里需要一个 PCO，II 型理论的全纯和反全纯扇区各需要一个 PCO——我们也无法找到一个在置换三个穿刺的 $SL(2, C)$ 变换下保持不变的单一位置。因此我们必须对不同 PCO 位置的选择取平均。

The second subtlety arises as follows. Let us suppose that we have chosen the $\mathcal{V}_{g,n}$ ' s so that they do not contain spurious poles. Now if we construct a Feynman diagram by joining two of them by a propagator, then it gives a particular choice of PCO locations determined by the plumbing fixture rules. One can now ask: is the configuration of PCOs determined this way free from spurious singularities? Unless the answer is in the affirmative, we run into a problem since in string field theory we do not have any freedom in choosing the PCO locations in part of $\widehat{\mathcal{P}}_{g,n}$ that is covered by Feynman diagrams with one or more propagators. They are fixed by the choice of PCO locations in the construction of the interaction vertices of the diagram. To answer this question, we need to examine the origin of spurious singularities in a Feynman diagram, assuming that interaction vertices have been chosen avoiding the spurious poles. For this it will be useful to represent the contribution from such Feynman diagrams as in quantum field theory. In this case we need to sum over infinite number of fields that could propagate along each internal propagator of the Feynman diagram, and this sum could diverge. One can show that this is the source of potential spurious poles in the Feynman diagram. This can be remedied by choosing the local coordinates at the punctures that we use in the construction of $\mathcal{V}_{g,n}$ to have long stubs in the sense described in section "Minimal Area String Vertices: Witten Vertex and Closed String Polyhedra". In this case, an external state of L_0 eigenvalue h attached to a vertex will produce a factor of λ^{-h} for some large number λ . Since for large h the number of states grow as $e^{C\sqrt{h}}$ for some constant C , we see that the sum over intermediate states in a Feynman diagram can be made convergent by taking λ to be large, and hence the Feynman diagrams with propagators will be free from spurious poles as long as the vertices $\mathcal{V}_{g,n}$ are constructed avoiding the spurious poles. For this argument, it is important to ensure that the spectrum of L_0 eigenvalues is bound from below and that for a given h , there are only a finite number of fields so that the sum is finite. The first condition is satisfied by choosing the states in the -1 picture in the NS sector and the $-1/2$ or $-3/2$ picture in the R sector. Both $-1/2$ and $-3/2$ picture R sector states have infinite degeneracies due to the existence of bosonic zero modes γ_0 and β_0 , respectively, causing apparent violation of the second condition. However, insertion of $\mathcal{X}_0$ in the R sector propagator guarantees that the matrix element of $\mathcal{X}_0$ between a pair of states is non-zero only for a finite number of states for any given L_0 eigenvalue [15]. This way, the second condition is also satisfied.

第二个微妙之处如下所述。假设我们已经选好了 $\mathcal{V}_{g,n}$ ，使其不包含伪极点。现在如果我们通过传播子连接两个 $\mathcal{V}_{g,n}$ 来构造费曼图，它会给出由 plumbing fixture 规则确定的 PCO 位置的特定选择。我们不禁要问: 这样确定的 PCO 构型是否不存在伪奇点？如果答案不是肯定的，我们就会遇到问题，因为在弦场论中，对于被一个或多个传播子的费曼图覆盖的 $\widehat{\mathcal{P}}_{g,n}$ 部分，我们完全没有自由选择 PCO 位置的余地——它们完全由图相互作用顶点构造中对 PCO 位置的选择固定。为了回答这个问题，我们需要在假设相互作用顶点已经避开伪极点选取的前提下，考察费曼图中伪奇点的来源。为此，将这类费曼图的贡献按照量子场论的方式表示会很有帮助。在这种情况下，我们需要对可以沿费曼图每个内部传播子传播的无穷多场求和，而这个求和可能发散。可以证明，这就是费曼图中潜在伪极点的来源。这个问题可以解决: 只要我们构造 $\mathcal{V}_{g,n}$ 时，在 puncture 处选取的局部坐标按照论文《极小面积弦顶点: 威顿顶点与闭弦多面体》中描述的意义带有长柄。在这种情况下，附着在顶点上、具有 L_0 本征值 h 的 L_0 外态会产生一个因子 λ^{-h} ，其中 λ 是一个大数。由于对于大的 h ，态的数目会按 $e^{C\sqrt{h}}$ 增长 (C 为常数)，我们可以看到，只要取足够大的 λ ，就能让费曼图中对中间态的求和收敛，因此只要顶点 $\mathcal{V}_{g,n}$ 本身构造时避开了伪极点，带传播子的费曼图就不会存在伪极点。对于这个论证，有一点很重要: 必须保证 L_0 本征值的谱有下界，且对给定的 h ，只有有限个场，因此求和是有限的。第一个条件可以通过在 NS 区选取-1 鬼图的态，在 R 区选取 $-1/2$ 或 $-3/2$ 鬼图的态来满足。R 区的 $-1/2$ 鬼图和 $-3/2$ 鬼图态都存在无穷简并，分别是因为存在玻色零模 γ_0 和 β_0 ，这看起来违反了第二个条件。但 R 区传播子中插入 $\mathcal{X}_0$ 可以保证: 对任意给定的 L_0 本征值，一对态之间 $\mathcal{X}_0$ 的矩阵元仅对有限个态非零 [15]。这样，第二个条件也得到了满足。

Superstring Field Theories in the Large Hilbert Space

大希尔伯特空间下的超弦场论

One of the issues we face in the construction of the interaction vertices for superstring field theories is in the choice of the locations of the picture changing operators. While the principle underlying the choice of picture changing operators is known, explicit constructions are more difficult. In this section, we shall describe a way to get around this issue by working in the large Hilbert space where we do not require the string field to be annihilated by η_0 . In that case, we can write down the gauge-invariant string field theory action in terms of the same products that appear in the construction of A_∞ or L_∞ algebras underlying the bosonic open or closed string field theories. The price we pay is that the covering of the moduli space by the sum over Feynman diagrams is no longer obvious; it requires extra effort to establish this.

我们在构建超弦场论相互作用顶点时面临的问题之一，是选择图变算子的位置。虽然选择图变算子的基本原理已经明晰，但具体构造仍难度较大。在本节中，我们将介绍一种规避该问题的方法: 在大希尔伯特空间中工作，该空间不要求弦场被 η_0 湮灭。在这种情况下，我们可以写出规范不变的弦场论作用量，所用的就是构造玻色开弦或闭弦场论基础的 A_∞ 或 L_∞ 代数时出现的乘积。我们付出的代价是，通过费曼图求和覆盖模空间的性质不再显然，需要额外的工作才能证明这一点。

All our discussion here will be restricted to the tree-level string field theories. We will consider classical open superstrings in the NS sector, classical heterotic strings in the NS sector, and classical type II strings in the NS-NS sector. We conclude with a brief discussion of the Ramond sector of classical open superstrings.

我们这里所有讨论都将仅限于树级弦场论。我们会考察 NS 区的经典开超弦、NS 区的经典杂化弦，以及 NS-NS 区的经典 II 型弦，最后简要讨论经典开超弦的拉蒙德区。

Berkovits Open Superstring Field Theory in the NS Sector

NS 领域的 Berkovits 开超弦场论

The conventional BRST quantization of the open superstring considers states $|\phi\rangle$ that are killed by the BRST operator: $Q\,|\phi\rangle = 0$. The familiar physical state | representatives in the NS sector take the schematic form $|\phi\rangle = ce^{-\phi}V_M\,(0)|\,0\rangle$, with V_M a matter operator that is a dimension one-half primary, $e^{-\phi}$ a dimension one-half primary, and c a dimension minus one primary, all for a dimension zero state. It follows that $|\phi\rangle$ has ghost number one and picture number minus one, the standard assignments of ghost and picture numbers for a NS string field. Correlators on the disk are nonvanishing unless the total picture number of the operators adds to minus two and the ghost numbers add up to three. This means that a kinetic term of the form $\langle\phi, Q\phi\rangle$ is allowed, but a cubic interaction of the string field requires the insertion of a PCO, an operator of ghost number zero and picture number plus one, to get the picture number to work out. As discussed in section ”String Vertices for Open Superstring Field Theory”, if one tries to use the associative open string vertex to build the string field action, this approach runs into some complications with collision of PCO’s, but using more general non-associative vertices, a classical theory can be formulated.

开超弦的常规 BRST 量子化考虑被 BRST 算子零化的态 $|\phi\rangle$:NS 领域中的 $Q\,|\phi\rangle = 0$. The familiar physical state | 代表具有概型形式 $|\phi\rangle = ce^{-\phi}V_M\,(0)|\,0\rangle$ ，其中 V_M 是维数为 1/2 的初态物质算子，$e^{-\phi}$ 是维数为 1/2 的初态，c 是维数为-1 的初态，整体组成一个维数为 0 的态。由此可知 $|\phi\rangle$ 的鬼数为 1，图片数为-1，这正是 NS 弦场的鬼数与图片数的标准设定。圆盘上的关联函数只有当所有算子的总图片数加起来为-2、总鬼数加起来为 3 时才不为零。这说明形如 $\langle\phi, Q\phi\rangle$ 的动力学项是允许的，但弦场的三次相互作用需要插入一个 PCO(图片变换算子，鬼数为 0、图片数为 +1 的算子) 才能满足图片数的要求。正如“开超弦场论的弦顶点”一节所述，如果尝试用结合开弦顶点构造弦场作用量，这种方法会遇到 PCO 碰撞的复杂问题，但使用更一般的非结合顶点可以构造出经典理论。

Now we discuss a different approach leading to an action without any explicit picture changing operators. This can be done by working in the ”large” Hilbert space of the β, γ conformal field theory. Recall that while ”fermionizing” the β, γ system via $\beta = \partial\xi e^{-\phi}$ and $\gamma = \eta e^{\phi}$, the field ξ appears under a derivative, and thus its zero mode ξ_0 does not feature in the construction of the β, γ fields. Working on the large Hilbert space amounts to including explicitly the anticommuting zero mode operator ξ_0 among the list of operators, thus immediately doubling the number of basis states. In this context, the conventional Hilbert space is called the ”small” Hilbert space, to distinguish it from the ”large” Hilbert space that includes states obtained by the action of ξ_0 on the small Hilbert space. We note that ξ_0 and the zero mode η_0 of the η field satisfy the anticommutator relation

现在我们讨论另一种可以得到不含任何显式图片变换算子的作用量的方法。这可以通过在 β, γ 共形场论的“大”希尔伯特空间中工作实现。回顾一下，当我们通过 $\beta = \partial\xi e^{-\phi}$ 和 $\gamma = \eta e^{\phi}$ 对 β, γ 系统“费米化”时，场 ξ 出现在导数下，因此它的零模 ξ_0 不会出现在 β, γ 场的构造中。在大希尔伯特空间中工作相当于将反对易零模算子 ξ_0 显式包含在算子列表中，直接将基态的数量翻倍。在此语境下，常规希尔伯特空间被称为“小”希尔伯特空间，以区别于包含通过 ξ_0 作用在小希尔伯特空间上得到的态的“大”希尔伯特空间。我们注意到 ξ_0 和 η 场的零模 η_0 满足反对易关系

$$\{\eta_0, \xi_0\} = 1 \tag{498}$$

The states in the small Hilbert space are those annihilated by η_0 . Of course, the large Hilbert space contains the small Hilbert space as a nontrivial subspace. The BRST operator Q and the η_0 operator are on similar footing in that $QQ = 0$ and $\eta_0\eta_0 = 0$. Moreover, they anticommute,

小希尔伯特空间中的态是被 η_0 零化的态。当然，大希尔伯特空间将小希尔伯特空间作为非平凡子空间包含在内。BRST 算子 Q 和 η_0 算子地位类似，满足 $QQ = 0$ 和 $\eta_0\eta_0 = 0$ 。此外，二者反对易，

$$\{Q, \eta_0\} = 0, \tag{499}$$

and both have ghost number one. In the following table, we indicate the conformal dimension, ghost number, and picture number of the various fields

且二者的鬼数均为 1。下表给出了各场的共形维数、鬼数和图片数

	dim	gh #	pic #
ξ	0	-1	+1
η	1	1	-1
β	3/2	-1	0
γ	-1/2	+1	0
$\exp(q\phi)$	$-\frac{1}{2}q(q+2)$	0	q

As we will describe now, the NS open string field Φ to be used in the large Hilbert space SFT is Grassmann even and has both ghost number zero and picture number zero. The construction of a fully gauge invariant interacting action is done in analogy to the construction of a Wess-Zumino-Witten theory of scalars on a two-dimensional compact space, nicely written when using complex coordinates z and $\bar{z}$. The role of the spatial derivatives ∂ and $\bar{\partial}$ is played in the string field theory by the operators Q and η_0 , respectively. The action is given by

我们接下来会说明，大希尔伯特空间弦场论中使用的 NS 开弦场 Φ 是格拉斯曼偶的，鬼数和图片数都为零。完全规范不变的相互作用量的构造类比了二维紧空间上标量的 Wess-Zumino-Witten 理论的构造，使用复坐标 z 和 $\bar{z}$ 可以得到简洁的表述。空间导数 ∂ 和 $\bar{\partial}$ 在弦场论中分别由算子 Q 和 η_0 扮演对应角色。作用量形式为

$$S = \frac{1}{2g^2} \left\| \left(e^{-\Phi}\eta_0 e^{\Phi}\right)\left(e^{-\Phi}Qe^{\Phi}\right) + \int_0^1 dt \Phi \left\{e^{-t\Phi}Qe^{t\Phi}, e^{-t\Phi}\eta_0 e^{t\Phi}\right\} \right\|. \tag{500}$$

This action is defined by expanding all exponentials in formal Taylor series and preserving the order of operators, letting the multilinear function $\langle\langle\cdots\rangle\rangle$ of an ordered set of operators be given by

该作用量的定义方式是将所有指数展开为形式泰勒级数并保留算序，有序算子集合的多重线性函数 $\langle\langle\cdots\rangle\rangle$ 由下式给出

$$\langle\langle A_1 \ldots A_n\rangle\rangle = \left\langle h^{-1} \circ f_1^{(n)} \circ A_1(0) \cdots h^{-1} \circ f_n^{(n)} \circ A_n(0)\right\rangle_L^{\prime},\ n \geq 2, \tag{501}$$

where the functions $f_k^{(n)}(z)$, prescribing the conformal maps to insert the operators, are

其中规定算子插入共形映射的函数 $f_k^{(n)}(z)$ 为

$$f_1^{(n)}(z) = \left(\frac{1+iz}{1-iz}\right)^{2/n},\ f_k^{(n)}(z) = e^{2\pi i(k-1)/n} f_1^{(n)}(z),\ k = 2, \ldots n, \tag{502}$$

and h^{-1} is the inverse of the map h defined in (470), taking the disk to the upper half plane. The subscript L on the correlator indicates that the upper half plane correlator in the definition of the multilinear product is computed in the large Hilbert space, where we have

其中 h^{-1} 是 (470) 中定义的映射 h 的逆，将圆盘映射至上半平面。关联函数上的下标 L 表示多线性乘积定义中的上半平面关联函数是在大希尔伯特空间中计算的，在此空间中我们有

$$\left\langle \xi c\partial c\partial^2 c e^{-2\phi}\right\rangle_L^{\prime} \neq 0, \tag{503}$$

telling us that for a nonvanishing correlator of a set of operators, the total ghost number must add up to two and the total picture number must add up to minus one. Note that each term in the action has just one Q and one η_0 . This requires a string field of zero picture number and zero ghost number, so that each term has picture number minus one, due to the η_0 , and ghost number two, due to Q and η_0 .

这告诉我们，对于一组非零的算符关联函数，总鬼数必须加起来为 2，总图数必须加起来为-1。注意作用量中的每一项都恰好包含一个 Q 和一个 η_0 。这要求弦场具有零图数和零鬼数，因此由于 η_0 ，每一项的图数为-1，由于 Q 和 η_0 ，每一项的鬼数为 2。

The function $f_1^{(n)}$ maps the upper half disk $|z| \leq 1, \operatorname{Im}(z) > 0$, to the wedge $\left|\operatorname{Arg}\left(f_1^{(n)}\right)\right| \leq \pi/n, \left|f_1^{(n)}\right| \leq 1$, of a full unit disk, with the puncture $z = 0$ mapped to $f_1^{(n)} = 1$. The maps with $k > 1$ are obtained by a simple rotation of the first wedge. All in all, n non-overlapping wedges, touching along radial lines, fully fill the unit disk. This unit disk is further mapped to the upper half plane by the map h^{-1} . For $n = 2$, the multilinear function is in fact the BPZ inner product. For $n = 3$, it is the multilinear product of the associative open string vertex (compare with (471)). For $n > 3$, this multilinear map can be thought as an iterated product using the associative open string three-vertex:

函数 $f_1^{(n)}$ 将上半圆 $|z| \leq 1, \mathrm{Im}(z) > 0$ 映射到单位整圆的楔形区域 $\left|\mathrm{Arg}\left(f_1^{(n)}\right)\right| \leq \pi/n, \left|f_1^{(n)}\right| \leq 1$，将穿刺点 $z = 0$ 映射到 $f_1^{(n)} = 1$。带 $k > 1$ 的映射可以通过对第一个楔形做简单旋转得到。总而言之，n 个互不重叠、沿径向相切的楔形区域恰好填满整个单位圆。这个单位圆再通过映射 h^{-1} 进一步映射到上半平面。对于 $n = 2$，多线性函数实际上就是 BPZ 内积。对于 $n = 3$，它就是结合开弦顶点的多线性乘积 (与 (471) 对比)。对于 $n > 3$，这个多线性映射可以看作是利用结合开弦三顶点的迭代乘积:

$$\langle\langle A_1 \dots A_n \rangle\rangle = \langle A_1, A_2 \star A_3 \star \cdots \star A_n \rangle'_L. \tag{504}$$

It is therefore cyclic.

因此它是循环的。

One can show that the equation of motion from this nonlinear action takes a surprisingly simple form:

可以证明，这个非线性作用量给出的运动方程具有出乎意料的简单形式:

$$\eta_0 \left(e^{-\Phi} Q e^{\Phi}\right) = 0, \tag{505}$$

where all products, obtained after expanding the exponentials, are to be regarded as the usual star product. Moreover, the action is invariant under gauge transformations with parameters Λ and Ω

其中展开指数后得到的所有乘积都视为普通的星乘积。此外，作用量在带参数 Λ 和 Ω 的规范变换下不变

$$\delta e^{\Phi} = (Q\Lambda)\, e^{\Phi} + e^{\Phi}\, (\eta_0 \Omega)\,. \tag{506}$$

The expansion of the action to cubic order (with Q and η_0 killing the constant term in the exponential) gives

将作用量展开到三次项 (其中 Q 和 η_0 消去了指数中的常数项)，可得

$$S = \frac{1}{2g^2} \left\langle \frac{1}{2} (Q\Phi)(\eta_0\Phi) + \frac{1}{6} (Q\Phi)\left(\Phi(\eta_0\Phi) - (\eta_0\Phi)\Phi\right) \right\rangle + \mathcal{O}\left(\Phi^3\right). \tag{507}$$

From the above expansion, we see that the kinetic term takes the form

从上述展开可以看出，动能项的形式为

$$S_{\text{kin}} \sim \langle (Q\Phi)(\eta_0\Phi) \rangle = \langle Q\Phi, \eta_0\Phi \rangle'_L, \tag{508}$$

with the bilinear function equal to the BPZ inner product. This means that the linearized equation of motion is

其中双线性函数等于 BPZ 内积。这意味着线性化运动方程为

$$Q\eta_0|\Phi\rangle = 0. \tag{509}$$

Consistent with the full gauge transformations (506), we have two linearized gauge invariances:

与完整规范变换 (506) 一致，我们得到两个线性化规范不变性:

$$\delta\Phi = Q\,|\Lambda\rangle,\ \delta\Phi = \eta_0|\,\Omega\rangle. \tag{510}$$

To analyze the free field equation, we expand the string field relative to the zero mode ξ_0 :

为了分析自由场方程，我们相对于零模 ξ_0 展开弦场:

$$|\Phi\rangle = |\phi'\rangle + \xi_0|\,\phi\rangle,\ \text{ with }\ \eta_0\,|\phi'\rangle = \eta_0|\,\phi\rangle = 0. \tag{511}$$

Here $|\Phi\rangle$ is in the large Hilbert space, while $|\,\phi\rangle$ and $|\phi'\rangle$ are in the small Hilbert space. The η_0 gauge symmetry is explicitly

此处 $|\Phi\rangle$ is in the large Hilbert space, while $|\,\phi\rangle$ 和 $|\phi'\rangle$ 属于小希尔伯特空间。η_0 规范对称性具体写为

$$\delta\,|\Phi\rangle = \eta_0|\,\Omega\rangle = \eta_0\,(|\omega\rangle + \xi_0|\,\omega'\rangle) = |\omega'\rangle. \tag{512}$$

This can be used to gauge away $|\phi'\rangle$ in $|\Phi\rangle$ so that we have $|\,\Phi\rangle\ =\ \xi_0|\phi\rangle$. The equation of motion then becomes

我们可以利用它将 $|\Phi\rangle$ so that we have $|\,\Phi\rangle = \xi_0|\phi\rangle$ 中的 $|\phi'\rangle$ 规范掉，此时运动方程变为

$$0 = Q\eta_0\,|\Phi\rangle = Q\eta_0\xi_0|\,\phi\rangle = Q\,\{\eta_0, \xi_0\}\,|\phi\rangle = Q|\,\phi\rangle. \tag{513}$$

Since $|\phi\rangle$ is in the small NS Hilbert space and satisfies the familiar linearized field equation, we conclude that $|\phi\rangle$ is a state of ghost number one and picture number | minus one. Given that the string field $|\Phi\rangle$ is obtained by acting with ξ_0 on $|\phi\rangle$ we have, as expected, that the large Hilbert space NS open string field Φ has ghost and picture number zero.

由于 $|\phi\rangle$ 属于 NS 小希尔伯特空间且满足我们熟悉的线性化场方程，我们可得 $|\phi\rangle$ is a state of ghost number one and picture number | 等于-1。鉴于弦场 $|\Phi\rangle$ 是通过将 ξ_0 作用在 $|\phi\rangle$ 上得到的，正如预期，大希尔伯特空间 NS 开弦场 Φ 的鬼数和图数均为零。

The NS sector open string field theory described here can be shown to be equivalent to the one described in section "String Vertices for Open Superstring Field Theory" after partial gauge fixing and appropriate field redefinition [144- 147]. BRST quantization of this theory has been discussed in [148].

可以证明，本文描述的 NS 区开弦场论，经过部分规范固定和适当的场重新定义后，等价于“开超弦场论的弦顶点”一节中描述的理论 [144-147]。该理论的 BRST 量子化已在文献 [148] 中讨论。

Heterotic String Field Theory in the NS Sector

NS 区的杂合弦场论

In this theory, the holomorphic sector is the NS sector of open superstrings, while the anti-holomorphic sector is that of a bosonic open string. We are thus effectively tensoring a string field that is Grassmann even and of ghost and picture number zero (NS large Hilbert space string field) with a string field that is Grassmann odd and of ghost number one (bosonic open string field). As a result, the heterotic string field V will be a Grassmann odd string field of ghost number one and picture number zero, anniihilated by b_0^- and L_0^- . In vertex operator language, V for physical states would be represented by $\xi c\bar{c}V_M e^{-\phi}$, where V_M is a matter sector primary operator of dimension $(1,1/2)$. This is clearly an operator carrying zero picture number and ghost number one.

在该理论中，全纯区是开超弦的 NS 区，反全纯区是玻色开弦的区。因此我们实际上将一个格拉斯曼偶、鬼数与图数均为零的弦场 (NS 大希尔伯特空间弦场)，与一个格拉斯曼奇、鬼数为 1 的弦场 (玻色开弦弦场) 做张量积。由此，杂合弦场 V 是格拉斯曼奇、鬼数为 1、图数为零的弦场，满足 b_0^- 和 L_0^- 湮灭条件。在顶点算符语言中，物理态的 V 可表示为 $\xi c\bar{c}V_M e^{-\phi}$ ，其中 V_M 是物质区的原初算符，维度为 $(1,1/2)$ 。显然该算符的图数为零、鬼数为 1。

The basic nonvanishing correlator in the large Hilbert space is now

大希尔伯特空间中基本非零关联函数为

$$\left\langle \xi c\partial c\partial^2 c\bar{c}\bar{\partial}\bar{c}\bar{\partial}^2\bar{c}e^{-2\phi}\right\rangle_L \neq 0, \tag{514}$$

showing that operators with a non-vanishing correlator must have total ghost number five and total picture minus one. Since the closed string inner product $\langle A,B\rangle_L$ includes a c_0^- factor, a non-vanishing inner product requires the ghost numbers of A and B to add up to four, and the picture numbers to add up to minus one.

这表明，存在非零关联函数的算符必须满足总鬼数为 5，总图数为-1。由于闭弦内积 $\langle A,B\rangle_L$ 包含一个 c_0^- 因子，非零内积要求 A 和 B 的鬼数之和为 4，图数之和为-1。

The tree-level action to cubic order in the string field V takes the form [25]

弦场 V 的树级作用量到三次项的形式为 [25]

$$S = 2\left[\frac{1}{2}\langle\eta_0 V, QV\rangle_L + \frac{\kappa}{3!}\langle\eta_0 V, [V, QV]_0\rangle_L\right] + \mathcal{O}\left(V^3\right), \tag{515}$$

where κ is related to g_s by a constant of proportionality that depends on the normalization in (514). In the above, the bracket $[,]_0$ is the genus zero contribution to the product. Together with the higher genus zero products appearing below, they represent any consistent solution of the L_∞ axioms in closed bosonic string theory, e.g., the products constructed using the minimal area metric or hyperbolic metric. The kinetic term implies a linearized equation of motion $\eta_0 QV = 0$ with linearized gauge invariances

其中 κ 与 g_s 通过一个比例常数关联，该常数依赖于 (514) 中的归一化。在上式中，括号 $[,]_0$ 是乘积的零亏格贡献。它和下文出现的其他零亏格乘积一起，构成闭玻色弦理论中 L_∞ 公理的任意自洽解，例如用最小面积度量或双曲度量构造的乘积。动能项给出线性化运动方程 $\eta_0 QV = 0$ ，对应线性化规范不变性为

$$\delta V = Q\Lambda + \eta_0 \Omega \tag{516}$$

The construction of the full nonlinear action and its gauge transformations is somewhat involved. A key ingredient is the generalization of the structure $A_Q \equiv e^{-\Phi} Q e^{\Phi}$ appearing in the open superstring action. The insight comes from the observation that A_Q is a pure gauge configuration in bosonic open string theory: it is of ghost number one; to leading order, it is $Q\Phi$; and it satisfies the cubic equation of motion $QA_Q + A_Q A_Q = 0$. The heterotic string field V of ghost number one and picture zero can play the role of a gauge parameter in bosonic closed string theory. Thus, QV is the leading term of a pure gauge configuration in closed string field theory. To find the full nonlinear expression $G(V)$ of the pure gauge, we introduce an integration parameter τ and the function $G(\tau V)$ that gives the desired $G(V)$ for $\tau = 1$ and vanishes for $\tau = 0$. Moreover, we require that $G(\tau V + d\tau V)$ and $G(\tau V)$ differ by a gauge transformation with parameter $d\tau V$ applied to $G(\tau V)$. Therefore,

完整非线性作用量及其规范变换的构造较为复杂。关键之处是推广开超弦作用量中出现的结构 $A_Q \equiv e^{-\Phi} Q e^{\Phi}$ 。我们的洞察来自于以下观察: A_Q 是玻色开弦理论中的纯规范构型: 它的鬼数为 1；领头阶为 $Q\Phi$ ；且满足三次运动方程 $QA_Q + A_Q A_Q = 0$ 。鬼数为 1、图数为零的杂合弦场 V 可以在玻色闭弦理论中充当规范参数。因此，QV 是闭弦场论中纯规范构型的领头项。为了得到纯规范的完整非线性表达式 $G(V)$ ，我们引入一个积分参数 τ 和函数 $G(\tau V)$ ，该函数在 $\tau = 1$ 时给出所需的 $G(V)$ ，在 $\tau = 0$ 时为零。此外我们要求，$G(\tau V + d\tau V)$ 和 $G(\tau V)$ 的差等于对 $G(\tau V)$ 应用带参数 $d\tau V$ 的规范变换。因此,

$$G(\tau V + d\tau V) = G(\tau V) + Q(d\tau V) + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} \left[G(\tau V)^n, d\tau V \right]_0 + \mathcal{O}\left(d\tau^2\right). \tag{517}$$

This gives the differential equation

由此得到微分方程

$$\partial_\tau G(\tau V) = QV + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} \left[G(\tau V)^n, V \right]_0. \tag{518}$$

This equation, written more schematically as

该方程可以更概要地写为

$$\partial_\tau G = QV + \kappa [G, V]_0 + \frac{\kappa^2}{2} [G, G, V]_0 + \mathcal{O}\left(\kappa^3\right), \tag{519}$$

can be solved in a power series expansion

可以通过幂级数展开求解

$$G = G^{(0)} + \kappa G^{(1)} + \kappa^2 G^{(2)} + \mathcal{O}\left(\kappa^3\right). \tag{520}$$

The first few terms are quickly calculated

前几项可以很快算出

$$G(V) = QV + \frac{\kappa}{2}[V, QV]_0 + \frac{\kappa^2}{3!}\left([V, QV, QV]_0 + \left[V, [V, QV]_0\right]_0\right) + \mathcal{O}\left(\kappa^3\right). \tag{521}$$

Since $G(V)$ is obtained from 0 by a series of infinitesimal gauge transformations, it should be a pure gauge configuration. This can be confirmed by showing that $G(V)$ is a solution of the bosonic closed string field theory equation of motion $F(\Psi) = 0$. This is done by forming a first-order linear differential equation for $F(G(\tau V))$, taking the form $\partial_\tau F(G(\tau V)) \sim F(G(\tau V))$. Since $G(\tau V)$ vanishes for $\tau = 0$, we also have $F(G(0)) = 0$, and by uniqueness, the solution of the differential equation is $F(G(\tau V)) = 0$. For $\tau = 1$, this is the claimed property.

由于 $G(V)$ 是通过一系列无穷小规范变换从 0 得到的，它应当是一个纯规范构型。这可以通过证明 $G(V)$ 是玻色闭弦场论运动方程 $F(\Psi) = 0$ 的解来确认。我们通过构造形如 $\partial_\tau F(G(\tau V)) \sim F(G(\tau V))$ 的 $F(G(\tau V))$ 一阶线性微分方程来完成该证明。由于当 $\tau = 0$ 时 $G(\tau V)$ 为零,我们还可得 $F(G(0)) = 0$ ，结合唯一性，该微分方程的解为 $F(G(\tau V)) = 0$ 。对于 $\tau = 1$ ，这就是我们所声称的性质。

The full nonlinear action can be written in a number of equivalent ways, useful for simple derivations of key properties. Perhaps the simplest form is as follows ([26], Section 5) [25]

完整的非线性作用量可以写成多种等价形式, 便于简单推导关键性质。最简单的形式大概如下 ([26], 第 5 节) [25]

$$S = 2\int_0^1 dt \langle \eta_0 V, G(tV)\rangle_L. \tag{522}$$

One can also determine the full nonlinear extensions of the linearized gauge transformations (516); for details see [26].

我们还可以确定线性化规范变换 (516) 的完整非线性推广，细节参见文献 [26]。

Type II String Field Theory in the NS-NS Sector

NS-NS sector 下的 II 型弦场论

For type II closed string field theory in the NS-NS sector, physical states in the large Hilbert space can be represented by operators of the form $\xi\bar{\xi}c\bar{c}e^{-\phi}e^{-\bar{\phi}}V_M$ that have left-moving and right-moving ghost numbers equal to zero as well as left-moving and right-moving picture numbers equal to zero. Experience with bosonic closed string theory suggests that the string field must have simply zero ghost number, without imposing two separate conditions; the inner product, for example, contains the factor c_0^- of indefinite left and right ghost numbers. On the other hand, it seems natural to fix separately the left and right picture numbers. Thus, the Grassmann even string field Ψ will have ghost number zero and left and right picture numbers zero.

对于 NS-NS sector 的 II 型闭弦场论，大希尔伯特空间中的物理态可以表示为 $\xi\bar{\xi}c\bar{c}e^{-\phi}e^{-\bar{\phi}}V_M$ 形式的算符，其左行、右行鬼数均为零，且左行、右行图数也均为零。玻色闭弦理论的研究经验表明，弦场仅需满足鬼数为零，无需额外施加两个独立条件；例如内积就包含左、右鬼数不确定的因子 c_0^- 。另一方面，分别固定左、右图数看起来是自然的。因此，格拉斯曼偶弦场 Ψ 的鬼数为零，且左、右图数均为零。

The basic nonvanishing correlator for type II theory in the large Hilbert space is

大希尔伯特空间中 II 型理论的基本非零关联函数为

$$\left\langle \xi\bar{\xi}c\partial c\partial^2 c\bar{c}\bar{\partial}\bar{c}\bar{\partial}^2\bar{c}e^{-2\phi}e^{-2\bar{\phi}}\right\rangle_L \neq 0, \tag{523}$$

which means that for a nonvanishing correlator of several operators, ghost numbers must add up to 4, and both left and right picture numbers must add up to minus one.

这意味着，对于多个算符构成的非零关联函数，总鬼数之和必须为 4，且左图数总和与右图数总和都必须为-1。

---

[25] A similarly simple form of the open string field theory action also exists and is given in [26]; Section 2.2.

[25] 开弦场论作用量也存在类似的简单形式，详见文献 [26] 第 2.2 节。

Therefore, in the bilinear form $\langle A, B\rangle$, ghost numbers of A and B must add up to three, and the left and right picture numbers must add up to minus one. The kinetic term for the theory consistent with these requirements is

因此，在双线性形式 $\langle A, B\rangle$ 中，A 和 B 的鬼数之和必须为 3，左图数与右图数之和必须为-1。满足这些要求的理论动能项为

$$S_2 = \frac{1}{2}\langle \eta_0\Psi, Q\bar{\eta}_0\Psi\rangle_L. \tag{524}$$

The linearized equation of motion is $\eta_0\bar{\eta}_0 Q\Psi = 0$ with linearized gauge invariances

线性化运动方程为 $\eta_0\bar{\eta}_0 Q\Psi = 0$ ，对应的线性化规范不变性为

$$\delta\Psi = Q\Lambda + \eta_0\Omega + \bar{\eta}_0\bar{\Omega}. \tag{525}$$

We can expand the string field $|\Psi\rangle = |\psi_0\rangle + \xi_0|\,\psi_1\rangle + \overline{\xi}_0\,|\psi_2\rangle + \xi_0\overline{\xi}_0\,|\psi_3\rangle$ with $|\psi_i\rangle$, with $i = 0, \ldots, 3$, in the small Hilbert space and thus annihilated by both η_0 and $\overline{\eta}_0$. The Ω gauge invariance can be used to set to zero $|\psi_0\rangle$ and $|\psi_2\rangle$. Then, the $\overline{\Omega}$ gauge invariance can be used to set $|\psi_1\rangle$ to zero. Thus, we have $\Big|\Psi\rangle = \xi_0\overline{\xi}_0\Big|\,\psi_3\rangle$, with $|\psi_3\rangle$ in the small Hilbert space and in the minus one, minus one picture. The linearized equation of motion then implies $Q\,|\psi_3\rangle = 0$, as required.

我们可以用 $|\psi_i\rangle$、$i = 0, \ldots, 3$ 对弦场 $|\Psi\rangle = |\psi_0\rangle + \xi_0|\,\psi_1\rangle + \overline{\xi}_0\,|\psi_2\rangle + \xi_0\overline{\xi}_0\,|\psi_3\rangle$ 做展开，展开后的弦场处于小希尔伯特空间中，因此会被 η_0 和 $\overline{\eta}_0$ 同时零化。利用 Ω 规范不变性可以将 $|\psi_0\rangle$ 和 $|\psi_2\rangle$ 置零，之后再利用 $\overline{\Omega}$ 规范不变性可以将 $|\psi_1\rangle$ 置零。由此我们得到 $\Big|\Psi\rangle = \xi_0\overline{\xi}_0\Big|\,\psi_3\rangle$，其中 $|\psi_3\rangle$ 处于小希尔伯特空间，且图数为 (-1,-1)。此时线性化运动方程自然给出满足要求的 $Q\,|\psi_3\rangle = 0$ 。

The cubic term of this theory begins to show some of the complexity. One finds:

该理论的三次项已经开始体现出一定的复杂性，可以得到:

$$S_3 = \frac{1}{3!}\Big\langle \eta_0\Psi, \big[Q\overline{\eta}_0\Psi, \overline{\eta}_0\Psi\big]^*\Big\rangle_L, \tag{526}$$

where the starred two-product must have an insertion of the PCO zero mode $\overline{\mathcal{X}}_0 = \Big\{Q, \overline{\xi}_0\Big\}$ to compensate for the presence of the two $\overline{\eta}_0$ factors. The $\overline{\mathcal{X}}_0$ is inserted on the three legs of the product:

其中带星号的二乘积必须插入 PCO 零模 $\overline{\mathcal{X}}_0 = \Big\{Q, \overline{\xi}_0\Big\}$，以抵消两个 $\overline{\eta}_0$ 因子带来的图数变化。$\overline{\mathcal{X}}_0$ 被插入在乘积的三个腿上:

$$[A, B]^* \equiv \frac{1}{3}\left(\overline{\mathcal{X}}_0[A, B]_0 + \left[\overline{\mathcal{X}}_0 A, B\right]_0 + \left[A, \overline{\mathcal{X}}_0 B\right]_0\right). \tag{527}$$

Since $\Big\{Q, \overline{\mathcal{X}}_0\Big\} = 0$, we see that Q is a derivation of the starred product. Since $\{\eta_0, \bar{\mathcal{X}}_0\} = \{\overline{\eta}_0, \bar{\mathcal{X}}_0\} = 0$, both η_0 and $\overline{\eta}_0$ are derivations of the starred product. A full type II action was built by Matsunaga [27] and takes the form

由 $\Big\{Q, \overline{\mathcal{X}}_0\Big\} = 0$ 可知，Q 是星乘积的一个导子。由 $\{\eta_0, \bar{\mathcal{X}}_0\} = \{\overline{\eta}_0, \bar{\mathcal{X}}_0\} = 0$ 可知，η_0 和 $\overline{\eta}_0$ 都是星乘积的导子。松永构建了完整的 II 型作用量 [27]，其形式为

$$S = \int_0^1 dt \langle \eta_0 \Psi_t, \mathcal{G}^*\left(\Psi(t)\right)\rangle_L. \tag{528}$$

Here, with $\Psi(t) = t\Psi$, we have that the NS-NS $\Psi_t = \Psi + \overline{\eta}_0\left(\cdots\right)$

在此，取 $\Psi(t) = t\Psi$ ，我们得到 NS-NS 扇区 $\Psi_t = \Psi + \overline{\eta}_0\left(\cdots\right)$

$$\Psi_t = \Psi + \frac{\kappa}{2} t\big[\overline{\eta}_0\Psi, \Psi\big]^* + \mathcal{O}\left(\kappa^2\right), \tag{529}$$

and

且

$$\mathcal{G}^* (\Psi) = Q\overline{\eta}_0 \Psi + \frac{\kappa}{2} t \big[Q\overline{\eta}_0 \Psi, \overline{\eta}_0 \Psi \big]^* + \mathcal{O}\left(\kappa^2\right). \tag{530}$$

The complete definition of $\mathcal{G}^* (\Psi)$ for the choice $\Psi(t) = t\Psi$ is as follows. Under an infinitesimal change $t \to t+\delta t$, $\mathcal{G}^* (\Psi(t+\delta t))$ is obtained from $\mathcal{G}^* (\Psi(t))$ by an infinitesimal gauge transformation with parameter $\delta t \overline{\eta}_0 \Psi$ in the heterotic string field theory formulated in the small Hilbert space in the anti-holomorphic superstring sector. Recall that the string field for the small Hilbert space heterotic string is of ghost number two and picture number minus one. Indeed $\left(\overline{\eta}_0 \Psi\right)$, playing the role of the gauge parameter in the above equation for $\mathcal{G}^*$, has ghost number one and picture number minus one. The computation of $\mathcal{G}^*$ uses the L_∞ -type heterotic action discussed in [28], with starred products bearing $\overline{\mathcal{X}}_0$ insertions in rather intricate combinatoric patters and satisfying the L_∞ constraint equations. The complete definition of Ψ_t is a bit more complicated but can be found in [27].

当选择 $\Psi(t) = t\Psi$ 时, $\mathcal{G}^* (\Psi)$ 的完整定义如下。在微小变换下, $t \to t+\delta t$, $\mathcal{G}^* (\Psi(t+\delta t))$ 可由 $\mathcal{G}^* (\Psi(t))$ 通过一个无穷小规范变换得到，该变换的参数 $\delta t \overline{\eta}_0 \Psi$ 属于定义在小希尔伯特空间、反全纯超弦扇区的杂化弦场论中。回顾可知，小希尔伯特空间杂化弦的弦场鬼数为 2，影数为-1。事实上，在上述 $\mathcal{G}^*$ 的方程中充当规范参数的 $\left(\overline{\eta}_0 \Psi\right)$ 鬼数为 1，影数为-1。$\mathcal{G}^*$ 的计算使用了文献 [28] 中讨论的 L_∞ 型杂化作用量，其中星乘积在相当复杂的组合模式中带有 $\overline{\mathcal{X}}_0$ 插入项，并且满足 L_∞ 约束方程。Ψ_t 的完整定义稍复杂一些，但可在文献 [27] 中找到。

Open Superstring Field Theory in the Ramond Sector

拉蒙域开超弦场论

So far the construction described in this section deals only with the NS sector of string field theory. At the classical level, this is a consistent truncation of the full string field theory. Nevertheless, even at the classical level, a complete construction of string field theory must include the Ramond sector. In this section, we shall describe an extension of the open superstring field theory described in section ”Berkovits Open Superstring Field Theory in the NS Sector” that includes the Ramond sector [29, 30] .

到目前为止，本节描述的构造仅涉及弦场论的 NS(诺伊堡-施瓦茨) 域。在经典层面，这是全弦场论的一致截断。但即便在经典层面，弦场论的完整构造也必须包含拉蒙 (Ramond) 域。本节我们将描述对“NS 域贝尔科维茨开超弦场论”一节中所述开超弦场论的推广，将拉蒙域纳入其中 [29, 30] 。

The NS sector of this string field theory is identical to the one described in section ”Berkovits Open Superstring Field Theory in the NS Sector”. For introducing the Ramond sector fields, we shall work in the small Hilbert space. Let us recall some basic facts about the R-sector. The (β, γ) ghosts admit mode expansions

该弦场论的 NS 域与“NS 域贝尔科维茨开超弦场论”一节中描述的完全一致。为引入拉蒙域场，我们将在小希尔伯特空间中开展工作。先来回顾 R 域的一些基本性质。(β, γ) 鬼允许模展开

$$\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-\frac{3}{2}}, \ \gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n+\frac{1}{2}}, \tag{531}$$

with the vacua $\left|-1/2\right\rangle = e^{-\phi/2}(0)\left|0\right\rangle, \left|-3/2\right\rangle = e^{-3\phi/2}(0)\left|0\right\rangle$ satisfying,

其中真空 $|-1/2\rangle = e^{-\phi/2}(0)|0\rangle, |-3/2\rangle = e^{-3\phi/2}(0)|0\rangle$ 满足:

$$\beta_n|-1/2\rangle = 0 \text{ for } n \geq 0,\ \gamma_n|-1/2\rangle = 0 \text{ for } n \geq 1. \tag{532}$$

We denote by G_0^m the zero mode of the matter sector superconformal generator and define

我们用 G_0^m 表示物质域超共形生成元的零模，并定义

$$X = -\delta(\beta_0)G_0^m + \delta'(\beta_0)b_0,\ Y = -c_0\delta'(\gamma_0). \tag{533}$$

The R-sector string field Ψ is taken to be a state of ghost number 1 and picture number $-1/2$ subject to the restriction that Ψ must be of the form

R 域弦场 Ψ 取为鬼数 1、鬼数 $-1/2$ 的态，且满足 Ψ 必须为如下形式的约束

$$\Psi = \phi - (\gamma_0 + 2c_0b_0\gamma_0 + c_0G_0^m)\psi, \tag{534}$$

with ϕ, ψ satisfying the constraints

其中 ϕ, ψ 满足约束条件

$$b_0\phi = 0,\ \beta_0\phi = 0,\ \eta_0\phi = 0,\ b_0\psi = 0,\ \beta_0\psi = 0,\ \eta_0\psi = 0. \tag{535}$$

This condition on Ψ may also be expressed as

Ψ 满足的这个条件也可以写作

$$XY\Psi = \Psi,\ \eta_0\Psi = 0, \tag{536}$$

where the second condition just expresses the fact that Ψ is in the small Hilbert space. We have written this condition explicitly since later we shall couple the R-sector fields to the NS sector fields, and the latter belong to the large Hilbert space.

其中第二个条件只是说明了 Ψ 处于小希尔伯特空间中。我们将该条件明确写出，是因为后续我们会将 R 域场与 NS 域场耦合，而后者属于大希尔伯特空间。

The quadratic term in the string field theory action in the R-sector is given by

R 域弦场论作用量中的二次项由下式给出:

$$S_R^{(0)} = -\frac{1}{2}\langle\Psi, YQ\Psi\rangle', \tag{537}$$

where the absence of the subscript L stands for the fact that the BPZ inner product is computed in the small Hilbert space. The action is invariant under the linearized gauge transformation,

其中没有下标 L 表示 BPZ 内积是在小希尔伯特空间中计算的。作用量在线性规范变换下不变:

$$\delta\,|\Psi\rangle = Q|\lambda\rangle, \tag{538}$$

where $|\lambda\rangle$ is a ghost number 1, picture number $-1/2$ operator, satisfying.

其中 $|\lambda\rangle$ 是鬼数为 1、图数为 $-1/2$ 的算符，满足。

$$XY\lambda = \lambda,\ \eta_0\lambda = 0. \tag{539}$$

The full interacting tree-level open string field theory involves the R-sector string field Ψ of ghost number one and picture number $-1/2$ in the small Hilbert space satisfying (536) and the NS sector string field Φ of ghost number zero and picture number zero in the large Hilbert space. The action is given by

完整的相互作用树级开弦场论包含: 满足 (536)、处于小希尔伯特空间、鬼数 1、图数 $-1/2$ 的 R 域弦场 Ψ ，以及处于大希尔伯特空间、鬼数 0、图数 0 的 NS 域弦场 Φ 。作用量由下式给出:

$$S = -\frac{1}{2}\langle\Psi, YQ\Psi\rangle' - \int_0^1 dt\Big\langle A_t\,(t)\,, QA_\eta\,(t) + (F\,(t)\,\Psi)^2\Big\rangle_L', \tag{540}$$

where

其中

$$A_\eta\,(t) = \eta e^{\Phi(t)} e^{-\Phi(t)},\ A_t\,(t) = \partial_t e^{\Phi(t)} e^{-\Phi(t)}, \tag{541}$$

and

且

$$F\,(t)\,\Psi = \Theta\,(\beta_0)\big\{A_\eta\,(t)\,, \Theta\,(\beta_0)\big\{A_\eta\,(t)\,, \cdots\Theta\,(\beta_0)\big\{A_\eta\,(t)\,, \Psi\big\}\cdots\big\}\big\}, \tag{542}$$

Θ denotes the Heaviside function and $\Phi\,(t)$ is subject to the boundary condition that $\Phi\,(1) = \Phi$. The products of the string fields appearing in this expression (including those obtained by expanding the exponentials) are to be interpreted as star products. One can show that the action (540) depends only on Ψ and the boundary value Φ of $\Phi\,(t)$. The full non-linear gauge symmetries of this action can be found in [29].

Θ 表示海维赛德函数，$\Phi\,(t)$ 满足边界条件 $\Phi\,(1) = \Phi$ 。该表达式中出现的弦场乘积 (包括指数展开得到的乘积) 都应理解为星乘积。可以证明，作用量 (540) 仅依赖于 Ψ 和 $\Phi\,(t)$ 的边界值 Φ 。该作用量完整的非线性规范对称性可在文献 [29] 中找到。

Note that the construction described above has two main ingredients: use of the Ramond sector string field in the restricted Hilbert space (536) and the construction of the gauge invariant coupling between the R-sector states and the NS sector states in the large Hilbert space. One could choose to use one and not the other and combine this with some other construction to write down gauge-invariant string field theory action.

This is precisely what was done in [139, 140] where the authors used the R-sector string fields in the restricted Hilbert space and the NS sector string fields in the small Hilbert space to write down a gauge-invariant string field theory action with cyclic A_∞ structure. Reference [149] has formulated a different version of open superstring field theory based on integrals over supermoduli space. We shall not discuss these developments in this review in any further detail.

请注意，上述构造包含两个核心要素: 在受限希尔伯特空间中使用拉蒙德扇区弦场 (536)，以及在大希尔伯特空间中构造 R 扇区态与 NS 扇区态之间的规范不变耦合。我们可以选择仅使用其中一个要素，再结合其他构造来写出规范不变的弦场论作用量。文献 [139, 140] 正是这么做的: 作者在受限希尔伯特空间中使用 R 扇区弦场，在小希尔伯特空间中使用 NS 扇区弦场，写出了具有循环 A_∞ 结构的规范不变弦场论作用量。文献 [149] 基于超模空间积分给出了开超弦场论的另一种表述。本综述不会对这些进展作进一步讨论。

Applications of String Field Theory

弦场论的应用

String field theory has had a number of applications that go beyond what the world-sheet formulation of string theory can achieve. In this section, we shall review some of these applications. Tachyon condensation, for example, involves non-perturbative physics. We will review this subject here, including the construction of the tachyon vacuum solution. There are a number of other important solutions of open string field theory that we will not be able to discuss here [150-156].

弦场论已拥有诸多超出弦理论世界面表述能力范围的应用。本节我们将回顾其中部分应用。例如，快子凝聚就涉及非微扰物理。我们将在此讨论这一主题，包括快子真空解的构造。还有许多其他重要的开弦场论解我们无法在此展开讨论 [150-156]。

We will also consider a number of topics in string perturbation theory that require string field theory to resolve a number of ambiguities in the world-sheet approach to the theory. Indeed, we will consider mass renormalization and vacuum shift. We will also review recent work on D-instanton contributions to closed string amplitudes. Finally, we will sketch the arguments for unitarity, crossing symmetry of string amplitudes, as well as ultraviolet finiteness of the theory.

我们还会探讨弦微扰论中的若干课题，这些课题需要借助弦场论解决世界面方法中的诸多歧义。具体来说，我们将讨论质量重整化和真空移位。我们也会回顾近期关于 D 瞬子对闭弦振幅贡献的研究工作。最后，我们将概略阐述弦振幅的幺正性、交叉对称性以及该理论的紫外有限性的相关论证。

There have been recent efforts to use string field theory to carry out conformal perturbation theory calculations that beyond leading orders become extremely subtle and possibly ambiguous. This is an important area of future research that we will not be able to cover here [87, 157-159].

近年已有研究尝试利用弦场论完成共形微扰论计算，这类计算在超出领头阶后会变得极为棘手，甚至可能存在歧义。这是未来研究的一个重要方向，我们无法在此展开讨论 [87, 157-159]。

With encouraging progress to date, it remains a goal to understand or improve string field theory to a degree that it gives a fully non-perturbative formulation of string theory.

尽管目前已取得令人振奋的进展，将弦场论理解或改进为能给出弦理论完整非微扰表述的形式仍然是一个有待实现的目标。

Tachyon Condensation

快子凝聚

Tachyon condensation is one of the first applications of string theory to problems that cannot be addressed using the standard world-sheet description of scattering amplitudes. Since by now there are plenty of reviews on this subject (see, e.g., [12] for analytic approaches and [14] for numerical approaches), we shall keep our discussion brief.

快子凝聚是弦理论最早应用于解决无法用散射振幅的标准世界面描述处理的问题的方向之一。由于目前关于该主题已有大量综述 (例如解析方法见文献 [12]，数值方法见文献 [14])，本文将简要讨论。

Open strings describe the excitations on the D-branes. Many D-branes in string theory have tachyonic excitation in the open string sector-excitations with negative mass-squared. This includes not only the D-branes in the bosonic string theory but also unstable D-brane systems in superstring theory, e.g., a coincident pair of a BPS brane and its anti-brane or non-BPS D-branes [160].

开弦描述 D 膜上的激发。弦理论中许多 D 膜的开弦扇区存在快子激发——即质量平方为负的激发。这不仅包括玻色弦理论中的 D 膜，也包括超弦理论中的不稳定 D 膜系统，例如一对重合的 BPS 膜与反 BPS 膜，或是非 BPS D 膜 [160]。

In perturbative quantum field theory, the existence of a tachyonic excitation is associated with the presence of a scalar field whose potential has a maximum at the background value of the field around which we carry out the perturbation expansion. If we study the theory by assuming that the fluctuations of the fields around this background remain small, we would conclude that there are tachyons in the spectrum. This, however, is not correct because the assumption that the fluctuations of the fields around the maximum of the potential can remain small is clearly wrong. The correct way to study such theories is to first identify a (local) minimum of the potential and then expand the potential around the minimum and quantize the theory. This will describe excitations with non-negative mass-squared. Therefore, in reality, there are no tachyonic modes in the spectrum.

在微扰量子场论中，快子激发的存在对应标量场的势在我们做微扰展开的背景场取值处取极大值。如果我们假设场在该背景附近的涨落始终很小来研究理论，会得出谱中存在快子的结论。但这并不正确，因为“势极大值附近的场涨落可以保持很小”这个假设显然不成立。研究这类理论的正确方法是先找到势的 (局部) 极小值，再围绕极小值展开势并对理论量子化。这样得到的激发质量平方都是非负的，因此实际上谱中不存在快子模式。

It is generally expected that open string tachyons on D-branes have similar origin, namely, that they are the result of attempting to quantize the system by expanding the potential around a maximum. However, in the world-sheet description of string theory, there is no systematic way to find the minimum of the potential and quantize the system by expanding the potential around the minimum. Nevertheless, using indirect arguments, one could make a guess about various properties of the minimum. These claims are the content of the following conjectures [57, 161]:

> 一般认为，D 膜上的开弦快子也源于类似的机制: 即它们是围绕势的极大值展开来对系统量子化得到的结果。然而，在弦理论的世界面描述中，没有系统的方法可以找到势的极小值并围绕极小值展开量子化系统。不过我们可以通过间接论证推测极小值的诸多性质，这些结论就是以下猜想的核心内容 [57, 161]:

1. The minimum of the tachyon potential on an unstable brane system, not protected by any conservation laws, describes a configuration where the brane disappears altogether, leaving just the closed string vacuum. Examples of such systems are D-branes in bosonic string theory and unstable D-branes or brane-anti-brane systems in superstring theory. One immediate consequence of this is that the difference in the height of the potential between the perturbative vacuum describing the original unstable brane system and the minimum of the potential must be equal to the tension of the original brane system.

> 1. 不受任何守恒律保护的不稳定膜系统中，快子势的极小值对应膜完全消失、仅留下闭弦真空的构型。这类系统的例子包括玻色弦理论中的 D 膜，以及超弦理论中的不稳定 D 膜或膜-反膜系统。由此可以直接推导出: 描述原不稳定膜系统的微扰真空与势极小值之间的势高度差，必定等于原膜系统的张力。

2. Since the minimum describes a configuration without any D-branes, there are no open string excitations around the minimum.

> 2. 由于极小值对应不存在任何 D 膜的构型，因此极小值附近没有开弦激发。

3. There are also non-trivial soliton solutions where the tachyon approaches the minimum of the potential asymptotically but takes a complicated form in the interior. These are expected to describe lower dimensional D-branes. Examples of such solutions are lump solutions in bosonic string D-branes describing lower-dimensional D-branes, a kink solution on a type II string non-BPS D-brane describing a BPS D-brane of one lower dimension, vortex solution on a brane-antibrane pair describing a BPS D-brane of two lower dimensions, etc.

> 3. 还存在非平庸孤子解: 这类解中快子渐近趋近势的极小值，但在内部呈现复杂形式。一般认为这类解描述低维 D 膜。这类解的例子包括: 玻色弦 D 膜中描述低维 D 膜的团解，II 型弦非 BPS D 膜上描述低一维 BPS D 膜的扭结解，膜-反膜对上描述低两维 BPS D 膜的涡旋解等等。

In special cases, the third conjecture may be studied using conformal field theory (CFT) techniques, since both the initial configuration and the final configuration, being D-branes, have a description in CFT. Studying the first two conjectures is harder since there is no good description of the tachyon vacuum in conformal field theory. [26] Open string field theory is well suited for studying all the conjectures since we can simply study classical solutions and their properties in string field theory.

在特殊情况下，由于初始构型和最终构型都是 D 膜，都可以用共形场论 (CFT) 描述，因此第三个猜想可以用共形场论技术研究。前两个猜想的研究难度更大，因为共形场论中没有对快子真空的良好描述。[26] 开弦场论非常适合研究所有这些猜想，因为我们可以直接在弦场论中研究经典解及其性质。

The main difficulty in studying the conjectures arises because the tachyon field is coupled to all the other fields, and so a systematic study requires studying classical equations of motion of a field theory with infinite number of fields. The initial study of this problem [167] was done using the "level truncation approach" a method first tried by Kostelecky and Samuel in open string field theory [168, 169] before the tachyon conjectures had been formulated. In level truncation, we truncate the string field to a finite set of fields based on the L_0 eigenvalue of the state that they multiply. We shall illustrate this with the example of unstable D-branes in bosonic string theory. At the leading order, we keep just the tachyon field $\phi c_1|0\rangle$ in the expansion of the string field since $c_1|0\rangle$ is the state of lowest L_0 eigenvalue. Also since we are looking for a translationally invariant solution, we keep only the zero momentum state. Substituting this into the cubic open string field theory action, one gets

研究这些猜想的主要难点在于快子场与所有其他场耦合，因此系统性研究需要对有无穷多个场的场论的经典运动方程进行研究。该问题的初始研究 [167] 采用了「水平截断方法」，该方法由 Kostelecky 和 Samuel 在快子猜想提出之前，率先在开弦场论中尝试 [168, 169]。在水平截断中，我们根据所乘态的 L_0 本征值，将弦场截断为有限组场。我们以玻色弦论中不稳定 D 膜为例说明。领头阶下，我们在弦场展开中仅保留快子场 $\phi c_1|0\rangle$，因为 $c_1|0\rangle$ 是 L_0 本征值最低的态。同时由于我们寻找平移不变解，我们只保留零动量态。将其代入三次开弦场论作用量，可得

$$-S = V g_o^{-2}\left[-\frac{1}{2}\phi^2 + \frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3 \phi^3\right], \tag{543}$$

where $V = (2\pi)^D \delta^{(p+1)}(0)$ is the space-time volume of the D-brane. The open string fields appearing in the cubic version of (139) have been scaled by a factor of g_o^{-1} to bring out an explicit factor of g_o^{-1} multiplying a g_o independent action. The $\left(3\sqrt{3}/4\right)^3$ factor comes from the conformal transformations needed to construct the cubic interaction vertex-these have been given in (150) and (151). On the other hand, as mentioned in (140) and proved in (256), the open string coupling g_o can be shown to be related to the D-brane tension $\mathcal{T}$ via the relation

其中 $V = (2\pi)^D \delta^{(p+1)}(0)$ 是 D 膜的时空体积。(139) 三次形式中的开弦场已按因子 g_o^{-1} 缩放，得到了一个显式的 g_o^{-1} 因子，它乘在一个与 g_o 无关的作用量上。$\left(3\sqrt{3}/4\right)^3$ 因子来自构造三次相互作用顶点所需的共形变换——相关内容已由 (150) 和 (151) 给出。另一方面，如 (140) 所述且已在 (256) 中证明，开弦耦合 g_o 可被证明与 D 膜张力 $\mathcal{T}$ 满足下述关系

$$\mathcal{T} = \frac{1}{2\pi^2 g_o^2}. \tag{544}$$

The potential in (543) has a local minimum at $\phi = \left(4/3\sqrt{3}\right)^3$, where it takes value

(543) 中的势在 $\phi = \left(4/3\sqrt{3}\right)^3$ 处存在局部极小，该处势的取值为

$$-S=-Vg_o^{-2}\frac{1}{6}\left(\frac{16}{27}\right)^3=-V\mathcal{T}2\pi^2\frac{1}{6}\left(\frac{16}{27}\right)^3\simeq-0.68V\mathcal{T}. \tag{545}$$

According to the first conjecture, the expected result is $-V\mathcal{T}$. So we are not very far from the expected result.

> 根据第一个猜想，预期结果为 $-V\mathcal{T}$ 。因此我们的结果与预期相差不远。

The analysis at higher order is facilitated using the observation that classical open string field theory admits a consistent truncation in which we set to zero all open string fields associated with excitations on non-trivial matter primary. Such states enter the open string field theory action in pairs and triplets but not singly, and it is therefore consistent to set them to zero. The truncated string field, containing excitations by matter Virasoro generators L_{-n}^m and ghost oscillators acting on the vacuum state, is known as the universal sector. Since the zero momentum tachyon belongs to the universal sector, we can try to extend the solution to higher levels remaining within the universal sector. A further simplification occurs by the use of "twist symmetry" that allows us to restrict to string fields multiplying states with odd L_0 eigenvalue, the tachyon being the one that multiplies a state of L_0 eigenvalue -1 . At the next order, working in the Siegel gauge, we include two more fields, describing the coefficient of $c_1L_{-2}^m|0\rangle$ and $c_{-1}|0\rangle$. One can construct the open string field theory action with this truncated string field and find the extremum of the action. The result improves dramatically, yielding about 95% of the expected answer [167]. One can also check that even though we construct the solution in the Siegel gauge, it also satisfies the equations of motion of the out of Siegel gauge fields to very good accuracy. This procedure has now been extended to very high level, yielding results very close to the expected result [170, 171]. This procedure has also been generalized to describe tachyon vacuum solution on unstable D-branes of type II string theory using Berkovits formulation of open superstring field theory [172, 173]. One surprising feature of the analysis based on level truncation is that there is no small parameter in which we carry out the expansion-a priori the contribution due to the higher level fields could be as large as that from the lower level fields. Yet level truncation seems to converge rapidly.

> 利用以下观测可以简化高阶分析: 经典开弦场论允许一致截断，我们可以将所有对应非平凡物质主激发的开弦场都置零。这类态以成对、成组的形式出现在开弦场论作用量中，不会单独出现，因此将其置零是自洽的。截断后的弦场包含物质 Virasoro 生成元 L_{-n}^m 和鬼振子作用在真空态上得到的激发，被称为通用区。由于零动量快子属于通用区，我们可以尝试在通用区内将解推广到更高水平。利用「扭转对称性」可进一步简化: 它允许我们仅保留乘了奇数 L_0 本征值态的弦场，而快子就是乘了 L_0 本征值为-1 的态的弦场。下一阶中，在 Siegel 规范下，我们额外加入两个场，分别描述 $c_1L_{-2}^m|0\rangle$ 和 $c_{-1}|0\rangle$ 的系数。我们可以构造该截断弦场对应的开弦场论作用量，并找到作用量的极值。结果得到了极大的改善，达到预期答案的约 95%[167]。我们还可以验证，即使我们在 Siegel 规范下构造解，它对出 Siegel 规范的场也能很好地满足运动方程。目前该方法已经被推广到非常高的水平，所得结果与预期非常接近 [170, 171]。该方法也被推广，用于在 II 型弦论的不稳定 D 膜上描述快子真空解，所用框架是 Berkovits 的开超弦场论表述 [172, 173]。基于水平截断的分析有一个出人意料的特点: 我们做展开时没有小参数——理论上高阶场的贡献可以和低阶场一样大，但水平截断依然收敛得很快。

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[26] Some understanding of this may be obtained using the boundary string field theory [162-164] formulated in [165,166]. We shall not discuss this here but refer the reader to the original literature.

[26] 我们可以利用在文献 [165,166] 中建立的边界弦场论 [162-164] 获得对该问题的一些理解。本文在此不做讨论，读者可参考原始文献。

One can also use a slightly modified version of this procedure to describe soliton solutions in open (super-) string field theory describing lower-dimensional D-branes [174]. Instead of working with a non-compact world-volume of the parent D-brane, we compactify certain directions that would eventually be transverse to the soliton. Then the momenta carried by the string field in these directions are quantized, and the expansion of string fields carrying momenta in these directions also admits a discrete level expansion, with higher level fields multiplying states of higher L_0 eigenvalues. We can now proceed as before to construct classical solutions in level expansion that describe lower-dimensional D-branes and check if the action associated with the solution accounts for the difference in the value of the action for the original D-brane and the lower-dimension D-branes that we want to construct. In all the cases that have been studied, the results come close to the expected values.

我们还可以使用该流程的一个稍经修改的版本，来描述开 (超) 弦场论中描述低维 D 膜的孤子解 [174]。我们不对母 D 膜的非紧致世界体积进行处理，而是对最终垂直于孤子的方向进行紧致化。此时弦场在这些方向上携带的动量被量子化，携带这些方向动量的弦场展开也可以得到离散的能级展开，更高能级的场对应更高 L_0 本征值的态。我们可以按照之前的步骤，在能级展开中构造描述低维 D 膜的经典解，验证该解对应的作用量是否能够解释原 D 膜和我们想要构造的低维 D 膜之间的作用量差值。在所有已研究的案例中，结果都和预期值吻合得很好。

Except for the leading order results, most of the analysis in the level truncation method has been carried out numerically, since analytical solution of algebraic equations of many variables is hard to find. The situation changed dramatically in 2005 when Martin Schnabl [175] wrote down an analytic solution in bosonic open string field theory describing the tachyon vacuum and showed analytically that the energy density of this solution exactly cancels the tension of the original D-brane. The solution was written down not in the Siegel gauge but in a different gauge that has come to be known as the Schnabl gauge. Since then different versions of the solution, related by gauge transformation to the original solution, have been constructed, and the algebraic structure underlying the construction of the solution has been understood [176, 177]. This has been reviewed in detail in [12]. We now also have a systematic procedure for describing any configuration of arbitrary number of static D-branes as a classical solution in the open string field theory on a specific D-brane as long as all of them share the same closed string background [153, 178]. The tools used are surface states, including the so-called wedge states, and the K, b, c algebra of string fields [176], both of which will be discussed below.

除领头阶结果外，能级截断方法中的大部分分析都是通过数值计算完成的，因为很难得到多变量代数方程的解析解。这种情况在 2005 年发生了巨大转变:Martin Schnabl[175] 在玻色开弦场论中写下了描述快子真空的解析解，并通过解析证明该解的能量密度恰好抵消原 D 膜的张力。这个解不是在 Siegel 规范下得到的，而是在一个后来被称为 Schnabl 规范的不同规范中得到的。此后人们构造了通过规范变换与原解关联的多种版本的解，也理解了该解构造背后的代数结构 [176,177]，相关内容已经在文献 [12] 中有详细综述。现在我们也已经有了一套系统流程: 只要所有 D 膜共享同一个闭弦背景，就可以将任意数量静态 D 膜的任意构型，描述为特定 D 膜上开弦场论的经典解 [153,178]。所用的工具是包含所谓楔形态在内的表面态， 以及弦场的 K, b, c 代数 [176]，这两者都会在下文讨论。

At present, we do not have a complete understanding of the relation between the Siegel gauge solution using level truncation and Schnabl's analytic solution. It will definitely be helpful to understand this relation better.

目前我们还没有完全理解使用能级截断得到的 Siegel 规范解和 Schnabl 解析解之间的关系。进一步理解这一关系无疑会大有帮助。

Tachyon Vacuum Solution and the $\mathcal{K}$ bc Algebra

快子真空解与 $\mathcal{K}$ bc 代数

We give here some of the basic ideas that go into the construction of classical open string field theory solutions, focusing on the tachyon vacuum solution. As mentioned above, this is just one of many solutions known at this point. For more details, consult the references above.

我们在此介绍经典开弦场论解构造的部分基本思路，重点关注快子真空解。如上所述，这只是目前已知的众多解之一，更多细节请参考上文所列参考文献。

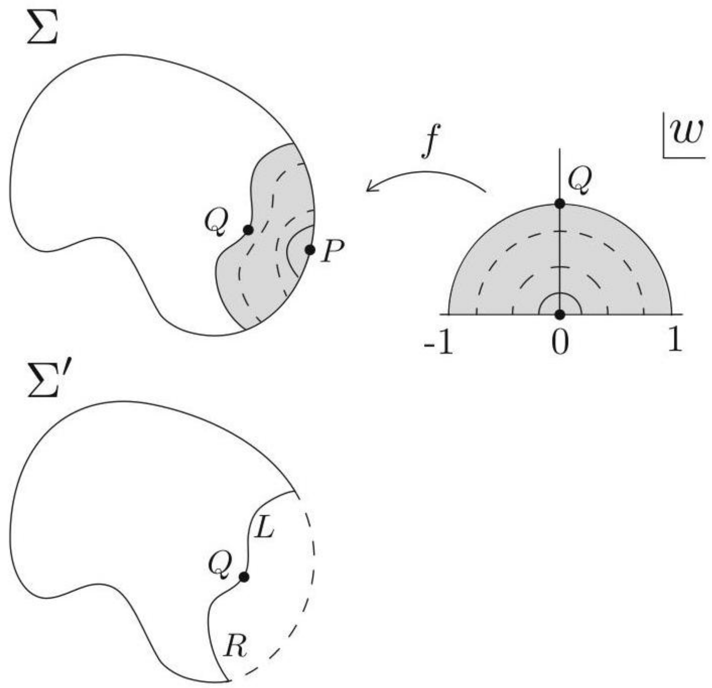
Solutions of OSFT are visualized as surface states dressed with some insertions of CFT operators. Surface states are the basic ingredient; they simply associate a surface with a puncture and a chosen local coordinate at the puncture to a state of the string. In our case, we have a surface $\sum$ that is topologically a disk, a chosen point P on the boundary of $\sum$, and a local coordinate at P , namely, a map f from a canonical half disk $\mathrm{Im}(w) > 0, |w| \leq 1$, to the surface $\sum$ equipped with a coordinate z . We have $z(P) = f(w=0)$, and the real boundary of the half disk is mapped to (part of) the boundary of $\sum$ (see Fig. 16). The surface state $\langle \sum |$ is defined through overlaps

开弦场论 (OSFT) 的解可以形象化为带有共形场论 (CFT) 算子插入的表面态。表面态是基础组成单元: 它只是将一个带孔的曲面、以及孔处选定的局部坐标，对应到弦的一个态。在我们的情形中，存在一个拓扑上为圆盘的曲面 $\sum$ ， $\sum$ 边界上的选定点 P ， 以及 P 处的局部坐标， 也就是一个从标准半圆盘 $\mathrm{Im}(w) > 0, |w| \leq 1$ 到配有坐标 z 的曲面 $\sum$ 的映射 f 。我们有 $z(P) = f(w=0)$ ， 半圆盘的实边界被映射到 $\sum$ 的 (部分) 边界 (参见图 16)。表面态 $\langle \sum |$ 通过重叠给出定义

$$\langle \sum | \phi \rangle' = \langle \phi(w=0) \rangle'_{\sum} = \langle f \circ \phi(0) \rangle'_{\sum}. \tag{546}$$

Fig. 16 A surface state $|\sum\rangle$ associated with the surface $\sum$ has a puncture P at the boundary and a local coordinate around it. This local coordinate is a map from the ξ upper-half disk to $\sum$. Shown below is the surface $\sum'$, with the coordinate patch removed, introducing a new boundary composed by the left and right parts (L and R) of an open string

图 16 与曲面 $\sum$ 关联的表面态 $|\sum\rangle$ 在边界处有一个孔 P， 以及孔周围的局部坐标。该局部坐标是从 ξ 上半圆盘到 $\sum$ 的映射。下图展示了去掉坐标补丁后的曲面 $\sum'$，由此产生了由开弦的左右部分 (L and R) 组成的新边界



The state $|\sum\rangle$ just follows from BPZ conjugation. The surface in the surface state is best visualized as $\sum'$, the surface defined as $\sum$ minus the image of the coordinate half disk. This removal induces a parameterized boundary, "the" open string. Of course, $\sum'$ still has part of the standard boundary of $\sum$, where the open string boundary conditions hold. The left (L) and right (R) parts of the open string, viewed from $\sum'$, are the images under f of $|w| = 1$ with $\arg(w) \in [0, \pi/2]$ and $\arg(w) \in [\pi/2, \pi]$, respectively. The open string midpoint is Q (Fig. 16).

态 $|\sum\rangle$ 可直接由 BPZ 共轭得到。表面态中的曲面最好形象化为 $\sum'$， 该曲面定义为 $\sum$ 去掉坐标半圆盘的像。这种移除产生了一个参数化边界，即“开弦”。当然，$\sum'$ 仍然保留了 $\sum$ 的一部分标准边界，开弦边界条件在该部分成立。从 $\sum'$ 来看，开弦的左 (L) 右 (R) 部分分别是 f 作用下带有 $\arg(w) \in [0, \pi/2]$ 和 $\arg(w) \in [\pi/2, \pi]$ 的 $|w| = 1$ 的像。开弦中点为 Q (图 16)。

The star multiplication of two surface states $|\sum_1\rangle$ and $|\sum_2\rangle$ is also a surface state $|\sum_1 \star \sum_2\rangle$. This surface $(\sum_1 \star \sum_2)'$, is obtained by gluing the R half-string in $\sum_1'$ to the L half-string of $\sum_2'$. In the surface $(\sum_1 \star \sum_2)'$, the L part of the string is that of $\sum_1'$, and the R part of the string is that of $\sum_2'$. With this rule, it is simple to visualize the star product of surface states. If one wishes, one can add to $(\sum_1 \star \sum_2)'$ a coordinate half-disk consistent with its L and R parts, in this way getting the full surface $\sum_1 \star \sum_2$.

两个表面态 $|\sum_1\rangle$ 和 $|\sum_2\rangle$ 的星乘积仍是一个表面态 $|\sum_1 \star \sum_2\rangle$。该曲面 $(\sum_1 \star \sum_2)'$ 由 $\sum_1'$ 的 R 半弦粘合到 $\sum_2'$ 的 L 半弦得到。在曲面 $(\sum_1 \star \sum_2)'$ 中，弦的 L 部分来自 $\sum_1'$，弦的 R 部分来自 $\sum_2'$。根据这个规则，我们可以很容易地直观理解表面态的星乘积。如果需要，我们可以给 $(\sum_1 \star \sum_2)'$ 添加一个和它的 L、R 部分相容的坐标半圆盘，由此得到完整曲面 $\sum_1 \star \sum_2$。

There is plenty of flexibility in defining surface states here: we must choose how to present the disk $\sum$ and how to specify the local coordinates. This can be illustrated with the example of ”wedge states,” surface states characterized by a real parameter $\alpha \geq 0$. The full surface, called $C_{\alpha+1}$, for a cylinder of circumference $\alpha+1$ is the region in the z UHP with $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \alpha + \frac{1}{2}$ (see Fig. 17). The vertical lines at $-\frac{1}{2}$ and $\alpha+\frac{1}{2}$ are identified via $z \sim z+\alpha+1$. Viewed as a disk, the ”center” is the point at $i\infty$, and the boundary is the real segment $z \in \left[-\frac{1}{2}, \frac{1}{2}+\alpha\right]$. The surface is, of course, a semi-infinite cylinder or just a cylinder. The puncture is at the origin $z=0$, and the local coordinate patch is the region $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$, a unit width vertical strip, where we can identify the L and R edges of the surface state as the vertical lines $\mathrm{Re}(z)=\frac{1}{2}$ and $\mathrm{Re}(z)=-\frac{1}{2}$, respectively. The local coordinate w on the strip is related to z as follows:

此处定义表面态有很大的灵活性: 我们需要选择如何展示圆盘 $\sum$，以及如何规定局部坐标。“楔形态”可以作为说明这个问题的例子，楔形态是由实参数 $\alpha \geq 0$ 表征的表面态。周长为 $\alpha+1$ 的圆柱的完整曲面称为 $C_{\alpha+1}$，是 z 上半平面 (UHP) 中满足 $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \alpha + \frac{1}{2}$ 的区域 (见图 17)。位于 $-\frac{1}{2}$ 和 $\alpha+\frac{1}{2}$ 的竖线通过 $z \sim z+\alpha+1$ 等同。若视为圆盘，“中心”是 $i\infty$ 处的点，边界是实线段 $z \in \left[-\frac{1}{2}, \frac{1}{2}+\alpha\right]$。当然这个曲面本身就是半无限圆柱，也可以就是普通圆柱。刺点位于原点 $z=0$，局部坐标邻域是区域 $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$，即单位宽度竖带，我们可以将该表面态的 L 边和 R 边分别对应为竖线 $\mathrm{Re}(z)=\frac{1}{2}$ 和 $\mathrm{Re}(z)=-\frac{1}{2}$。竖带上的局部坐标 w 与 z 的关系如下:

$$z=\frac{2}{\pi}\arctan w. \tag{547}$$

This indeed maps the w upper-half disk to the unit width vertical strip, also sending $w=0$ to $z=0$, the position of the puncture, and $w=i$ to $z=i\infty$, the position of the open string mid-point. This unit-strip picture of the half disk is sometimes called the sliver frame. The surface Ω_α' is then just the region of width α to the right of the coordinate strip. Its left boundary is the half-string L , and by the identification, its right boundary is the half string R . This is the wedge state Ω_α' or sometimes simply called Ω_α (Fig. 17).

这确实能将 w 上半圆盘映射为单位宽度竖带，同时将 $w=0$ 映射到刺点位置 $z=0$，将 $w=i$ 映射到开弦中点位置 $z=i\infty$。这种半圆盘的单位竖带表示有时被称为银框架。曲面 Ω_α' 就是坐标带右侧宽度为 α 的区域。它的左边界是半弦 L，由等同关系可知，它的右边界是半弦 R。这就是楔形态 Ω_α'，有时也直接称为 Ω_α (图 17)。

It should now be clear that we have the star product [179]

现在我们可以清楚得到星乘积 [179]

$$|\Omega_\alpha\rangle \star |\Omega_\beta\rangle = |\Omega_{\alpha+\beta}\rangle, \tag{548}$$

since gluing a region of width α to a region of width β gives a region $\sum'$ of width $\alpha+\beta$. Note that the full surface $\sum$, obtained by adding the unit-width coordinate strip, is a cylinder of circumference $\alpha+1$. A few wedge states are easily recognized. The wedge state with $\alpha = 0$ is the identity string field $\mathcal{I}$; this is clear because the surface

因为将宽度为 α 的区域与宽度为 β 的区域粘合，就能得到宽度为 $\alpha+\ \beta$ 的区域 $\sum'$ 。注意，加上单位宽度坐标带后得到的完整曲面 $\sum$ 是周长为 $\alpha+1$ 的圆柱。我们很容易识别出几种特殊的楔形态: 参数满足 $\alpha = 0$ 的楔形态就是单位弦场 $\mathcal{I}$ ，从曲面的性质就能看出这一点;

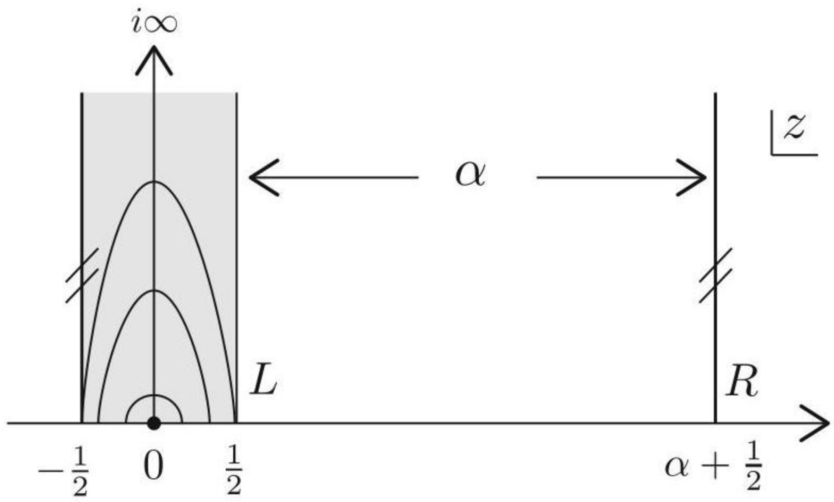


Fig. 17 The wedge state Ω_α . The coordinate patch is shown shaded and the leftmost and rightmost boundaries are identified to form the cylinder $C_{\alpha+1}$. The state Ω'_α has the coordinate patch removed and has boundaries L and R , for the left- and right halves of the open string

图 17 楔形态 Ω_α 。坐标邻域显示为阴影，最左和最右边界等同后形成圆柱 $C_{\alpha+1}$ 。去掉坐标邻域的态 Ω'_α 的边界分别为 L 和 R ，对应开弦的左半部分和右半部分

Ω'_0 is just a strip of vanishing length, which changes no state by star multiplication. The wedge state for $\alpha = 1$ is actually the $\mathrm{SL}(2,\mathbb{R})$ vacuum $|\Omega_1\rangle = |0\rangle$; this can be seen by mapping the region $-1/2 \leq \mathrm{Re}(z) \leq 3/2$, with the identification $z \equiv z+2$, to the full upper half w plane via the map (547). Finally, the state obtained in the limit $\alpha = \infty$ turns out to be well-defined, and it is called the sliver state $|\Omega_\infty\rangle$. The sliver is a rank one projector of the star algebra, satisfying the expected property $|\Omega_\infty\rangle \star |\Omega_\infty\rangle = |\Omega_\infty\rangle$. A few references that discuss manipulations involving wedge states and related surface states, with special emphasis on projectors, are [180-184].

Ω'_0 就是长度为零的带状区域，星乘不会改变任何状态。$\alpha = 1$ 的楔形态实际上就是 $\mathrm{SL}(2,\mathbb{R})$ 真空 $|\Omega_1\rangle = |0\rangle$ ；通过映射 (547) 可将满足等价关系 $z \equiv z+2$ 的区域 $-1/2 \leq \mathrm{Re}(z) \leq 3/2$ 映射到整个上半 w 平面，由此就能看出这一点。最后，极限 $\alpha = \infty$ 下得到的态是良好定义的，称为裂片态 $|\Omega_\infty\rangle$ 。裂片态是星代数的秩一投影子，满足预期性质 $|\Omega_\infty\rangle \star |\Omega_\infty\rangle = |\Omega_\infty\rangle$ 。讨论楔形态和相关表面态操作 (尤其着重投影子) 的部分文献可参见 [180-184]。

The next step in this construction is to understand how to change the width of a wedge state via the action of an operator. For this, consider an arbitrary [27] local boundary operator ϕ and the overlap

该构造的下一步是理解如何通过算符作用改变楔形态的宽度。为此，考虑任意一个 [27] 局部边界算符 ϕ 及其重叠

$$\langle \phi, \Omega_\alpha \rangle' = \langle f \circ \phi(0) \rangle'_{C_{\alpha+1}}, \tag{549}$$

where $f(w)$ is the map appearing on the right-hand side of (547). We evaluate the right-hand side by first doing a conformal scaling transformation $z \to f_\alpha(z) \equiv z/(\alpha+1)$ that turns the surface into a cylinder of unit circumference

其中 $f(w)$ 是 (547) 右侧出现的映射。我们先做共形标度变换 $z \to f_\alpha(z) \equiv z/(\alpha+1)$ 来计算右侧，这个变换将曲面转化为周长为单位 1 的圆柱

$$\langle \phi, \Omega_\alpha \rangle' = \langle f_\alpha \circ f \circ \phi(0) \rangle'_{C_1}. \tag{550}$$

Now note that $f_{\alpha+\delta\alpha}$, a scaling by $1/(1+\alpha+\delta\alpha)$, can be represented as the composition

现在注意，$f_{\alpha+\delta\alpha}$ ，即按 $1/(1+\alpha+\delta\alpha)$ 的标度，可以表示为复合变换

$$f_{\alpha+\delta\alpha} = f_{\delta\alpha/(1+\alpha)} \circ f_\alpha. \tag{551}$$

Applying (550) with α replaced by $\alpha+\delta\alpha$ we find

将 (550) 中的 α 替换为 $\alpha+\delta\alpha$ ，我们得到

$$\langle \phi, \Omega_{\alpha+\delta\alpha} \rangle' = \left\langle f_{\delta\alpha/(1+\alpha)} \circ f_\alpha \circ f \circ \phi(0) \right\rangle'_{C_1}. \tag{552}$$

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[27] We follow the discussion in [12], with a small modification that does not require the operator ϕ to carry definite conformal weight.

[27] 我们沿用 [12] 中的讨论，仅做了微小修改，不要求算符 ϕ 具有确定的共形权重。

The effect of the infinitesimal scaling $f_{\delta\alpha/(1+\alpha)}$ can be reproduced by the insertion of the operator $I - (1+\alpha)^{-1}\delta\alpha \oint_0 \frac{dz}{2\pi i} zT(z)$, with I the identity operator. Thus, we have, using (550)

无穷小标度 $f_{\delta\alpha/(1+\alpha)}$ 的效果可以通过插入算符 $I - (1+\alpha)^{-1}\delta\alpha \oint_0 \frac{dz}{2\pi i} zT(z)$ 重现，其中 I 是恒等算符。因此，利用 (550) 我们得到

$$\langle \phi, \Omega_{\alpha+\delta\alpha} \rangle' = \langle \phi, \Omega_\alpha \rangle' - \frac{\delta\alpha}{1+\alpha} \left\langle \oint_0 \frac{dz}{2\pi i} zT(z) f_\alpha \circ f \circ \phi(0) \right\rangle'_{C_1} \tag{553}$$

$$= \langle \phi, \Omega_\alpha \rangle' - \frac{\delta\alpha}{1+\alpha} \left\langle \oint_0 \frac{dz}{2\pi i} z T(z) \, f \circ \phi(0) \right\rangle'_{C_{\alpha+1}},$$

where in the last step, we have rescaled the cylinder width back to $(1+\alpha)$, noting that $\oint_0 \frac{dz}{2\pi i} z T(z)$ is invariant under this rescaling. We thus conclude that

其中最后一步，我们将圆柱宽度重新标度回 $(1+\alpha)$ ，注意到 $\oint_0 \frac{dz}{2\pi i} z T(z)$ 在该标度变换下不变。因此我们得出结论

$$\left\langle \phi, \frac{d}{d\alpha} \Omega_\alpha \right\rangle' = -\frac{1}{\alpha+1} \left\langle \oint_0 \frac{dz}{2\pi i} z T(z) \, f \circ \phi(0) \right\rangle'_{C_{\alpha+1}}. \tag{554}$$

At this point, by contour deformation, the integral around the vertex operator is deformed into the sum of two integrals, one on the boundary $\mathrm{Re}\, z = \alpha + \frac{1}{2}$ going up and one on $\mathrm{Re}\, z = -\frac{1}{2}$ going down. Of course, these edges are identified, so the integrals can be combined into a single integral, say, on the R boundary $\mathrm{Re}\, z = \alpha + \frac{1}{2}$. A short calculation gives

至此，通过围道变形，顶点算符周围的积分可变形为两个积分之和，一个沿 $\mathrm{Re}\, z = \alpha + \frac{1}{2}$ 边界向上，一个沿 $\mathrm{Re}\, z = -\frac{1}{2}$ 边界向下。这些边是等价认同的，因此积分可以合并为单个积分，例如在 R 边界 $\mathrm{Re}\, z = \alpha + \frac{1}{2}$ 上的积分。简短计算可得

$$\left\langle \phi, \frac{d}{d\alpha} \Omega_\alpha \right\rangle' = -\left\langle \int_R \frac{dz}{2\pi i} T(z) \, f \circ \phi(0) \right\rangle'_{C_{\alpha+1}}. \tag{555}$$

The right-hand side is in fact the overlap of $\langle \phi |$ with a wedge state acted by the contour integral of T . We write this as

右侧实际上就是 $\langle \phi |$ 与被 T 围道积分作用后的楔形态的重叠，我们将其写为

$$\frac{d}{d\alpha} \Omega_\alpha = -\Omega_\alpha \mathcal{K} = -\mathcal{K} \Omega_\alpha \tag{556}$$

where the last equality follows because the insertion could be contour-deformed within the wedge to the left line $\mathrm{Re}\, z = \frac{1}{2}$. Here $\mathcal{K}$ is a string field, a state, and the products in the above equation are star products. The state $\mathcal{K}$ is the surface state of an infinitesimally thin wedge with the insertion of the stress tensor or, equivalently, the identity string field $\mathcal{I}$ acted by the contour integral $\int \frac{dz}{2\pi i} T(z)$ over half the string, written schematically as

其中最后一个等式成立，因为插入项可以在楔内通过围道形变到左线 $\mathrm{Re}\, z = \frac{1}{2}$ 。此处 $\mathcal{K}$ 是一个弦场，即一个态，上述方程中的乘积都是星乘积。态 $\mathcal{K}$ 是无穷薄楔的表面态，带有能量动量张量的插入，等价地说，是半弦上的围道积分 $\int \frac{dz}{2\pi i} T(z)$ 作用在恒等弦场 $\mathcal{I}$ 上，形式化写为

$$\mathcal{K} = \left(\int_L T \right) \mathcal{I}. \tag{557}$$

Since $\Omega_0 = \mathcal{I}$, the above equation is integrated to find

由于 $\Omega_0 = \mathcal{I}$ ，对上述方程积分可得

$$\Omega_\alpha = e^{-\alpha\mathcal{K}}. \tag{558}$$

The exponential here is defined through string products $\left(e^X = \mathcal{I} + X + \frac{1}{2}X \star X + \cdots\right)$. This representation of the wedge states allows us to manipulate them effectively, as we shall see below.

此处的指数由弦乘积 $\left(e^X = \mathcal{I} + X + \frac{1}{2}X \star X + \cdots\right)$ 定义。我们将会看到，楔态的这种表示让我们可以有效地对其进行操作。

Completely analogous to $\mathcal{K}$, we have the state B , defined using the antighost field $b(z)$, instead of $T(z)$:

与 $\mathcal{K}$ 完全类似，我们得到态 B ，它是用反鬼场 $b(z)$ 而非 $T(z)$ 定义的:

$$B = \left(\int_L b\right)\mathcal{I}. \tag{559}$$

Finally we have the state c , defined an infinitesimal strip with a simple pointwise insertion of the operator $c(z)$ at the boundary (the base of the wedge). The $\mathcal{K}$ bc algebra is the set of identities satisfied by the string fields c, B , and $\mathcal{K}$. Including also the action of the BRST operator Q , we have the relations

最后我们得到态 c ，它定义在一个无穷小长条上，在边界 (楔的底边) 逐点简单插入算符 $c(z)$ 。$\mathcal{K}$ bc 代数就是弦场 c, B 和 $\mathcal{K}$ 满足的一组恒等式。加入 BRST 算符 Q 的作用后，我们得到关系

$$c^2 = 0,\ B^2 = 0,\ \{c, B\} = 1,$$

$$[\mathcal{K}, B] = 0,\ [\mathcal{K}, c] = \partial c, \tag{560}$$

$$Q\mathcal{K} = 0,\ QB = \mathcal{K},\ Qc = c\mathcal{K}c.$$

It turns out it is relatively easy to write solutions of the open string field theory using these states. It is harder, however, to tell what the physics of the solution is. Take, for example, the identity-based string field $\Psi = c(1-\mathcal{K})$ [185,186]. We see that $Q\Psi = c\mathcal{K}c - c\mathcal{K}c\mathcal{K} = -(c - c\mathcal{K})(c - c\mathcal{K}) = -\Psi^2$, thus Ψ is a solution. The BRST operator at the solution, $Q_\Psi = Q + [\Psi, \cdot]$, actually satisfies $Q_\Psi B = QB + \{c(1-\mathcal{K}), B\} = \mathcal{K} + \{c, B\} - \{c, B\}\mathcal{K} = \mathcal{I}$. An operator A satisfying $Q_\Psi A = \mathcal{I}$ is called a homotopy operator. Its existence implies that any BRST closed state χ is trivial: if $Q_\Psi \chi = 0$, we have $\chi = Q_\Psi (A \star \chi)$. For this solution, B is a homotopy operator, and therefore there are no physical states at the resulting configuration. The solution represents the tachyon vacuum, but regrettably, it is quite singular; the evaluation of the action for this solution is ill-defined.

事实证明, 利用这些态写出开弦场论的解相对容易, 但要确定这个解对应的物理内容却更困难。例如基于恒等的弦场 $\Psi = c(1-\mathcal{K})$ [185,186], 我们可以看到 $Q\Psi = c\mathcal{K}c - c\mathcal{K}c\mathcal{K} = -(c - c\mathcal{K})(c - c\mathcal{K}) = -\Psi^2$, 因此 Ψ 是一个解。该解处的 BRST 算符 $Q_\Psi = Q + [\Psi, \cdot]$ 实际上满足 $Q_\Psi B = QB + \{c(1-\mathcal{K}), B\} = \mathcal{K} + \{c, B\} - \{c, B\}\mathcal{K} = \mathcal{I}$ 。满足 $Q_\Psi A = \mathcal{I}$ 的算符 A 被称为同伦算符。它的存在意味着任意 BRST 闭态 χ 都是平凡的: 若 $Q_\Psi \chi = 0$, 则有 $\chi = Q_\Psi(A \star \chi)$ 。对于这个解, B 就是同伦算符, 因此所得构型中不存在物理态。这个解代表快子真空, 但遗憾的是它具有很强的奇异性; 这个解的作用量求值是不适定的。

A general solution found by Okawa [176] is written for general $F = F(\mathcal{K})$:

Okawa[176] 找到的通解对任意 $F = F(\mathcal{K})$ 写为:

$$\Psi = Fc\frac{\mathcal{K}}{1-F^2}BcF, \quad \text{with} \quad F = F(\mathcal{K}). \tag{561}$$

This can be checked with simple computations. First, one notices that

这可以通过简单计算验证。首先, 我们注意到

$$(1 - FBcF)^{-1} = 1 + \frac{F}{1-F^2}BcF, \tag{562}$$

which is verified by checking that the right-hand side multiplied by $(1 - FBcF)$ is indeed the identity (this requires using $BcBc = Bc$ and $BcFBc = FBc$). Then one checks that

可以验证, 右手边乘上 $(1 - FBcF)$ 确实得到恒等式 (这需要用到 $BcBc = Bc$ 和 $BcFBc = FBc$), 由此即可证明上式。接下来可以验证

$$\Psi = (1 - FBcF)\, Q(1 - FBcF)^{-1}. \tag{563}$$

In this form, where $\Psi = \xi Q \xi^{-1}$, with $\xi = 1 - FBcF$, it is essentially obvious that the equation of motion is satisfied: $Q\Psi = (Q\xi)(Q\xi^{-1}) = -\Psi^2$ because $\Psi^2 = (\xi Q\xi^{-1})(\xi Q\xi^{-1}) = -\xi\left[(Q\xi^{-1})\xi\right]Q\xi^{-1} = \xi\left[-\xi^{-1}Q\xi\right]Q\xi^{-1} = (Q\xi)(Q\xi^{-1})$. With a bit more work, one can check that there is a homotopy operator

在此形式中, 其中 $\Psi = \xi Q\xi^{-1}$, 满足 $\xi = 1 - FBcF$, 运动方程成立是显然的: $Q\Psi = (Q\xi)(Q\xi^{-1}) = -\Psi^2$, 因为 $\Psi^2 = (\xi Q\xi^{-1})(\xi Q\xi^{-1}) = -\xi\left[(Q\xi^{-1})\xi\right]Q\xi^{-1} = \xi\left[-\xi^{-1}Q\xi\right]Q\xi^{-1} = (Q\xi)(Q\xi^{-1})$. 。再稍加推导就能证明同伦算子存在

$$A = \frac{1-F^2}{\mathcal{K}}B, \; Q_\Psi A = \mathcal{I}. \tag{564}$$

Thus, formally, all these solutions are the tachyon vacuum. The question becomes: What are the allowed $F(\mathcal{K})$ for which the solution and the homotopy operator are both well-defined? There are two options that are particularly interesting [187]:

因此形式上来说, 所有这些解都是快子真空。问题变成: 什么样的允许 $F(\mathcal{K})$ 能让解和同伦算子都有良好定义? 有两种特别受关注的情况 [187]:

1. $F(\mathcal{K}) = e^{-\mathcal{K}/2} = \Omega_{1/2}$.

This corresponds to the original solution by Schnabl and gives

这对应施纳布尔的原始解，给出

$$\Psi = \Omega_{1/2} c \frac{\mathcal{K}}{1-\Omega_1} Bc\Omega_{1/2} = \Omega_{1/2} c (1 + \Omega_1 + \Omega_2 + \cdots) \mathcal{K} Bc \Omega_{1/2},$$

$$= \Omega_{1/2} c\mathcal{K} Bc\Omega_{1/2} + \Omega_{1/2} c \left(\sum_{n=1}^{\infty} \Omega_n \right) \mathcal{K} Bc\Omega_{1/2},$$

(565)

the last expression representing the solution as a leading ghost term supplemented by an infinite sum of wedge states of growing width, each with two insertions of c and one of B (Fig. 18, left). In this solution, the homotopy operator is believed to be well-defined. The solution itself, however, has a subtlety: the infinite sum cannot be truncated to evaluate the energy, and, as a result a so-called phantom term has to be added to the solution.

最后一个表达式将解写为首项鬼项，加上无限个宽度递增的楔形态之和，每个楔形态包含两次 c 插入和一次 B 插入 (图 18 左)。目前认为该解的同伦算子是良好定义的，但解本身存在一个微妙问题: 无限和无法截断来计算能量，因此需要给解额外添加一个所谓的幻项。

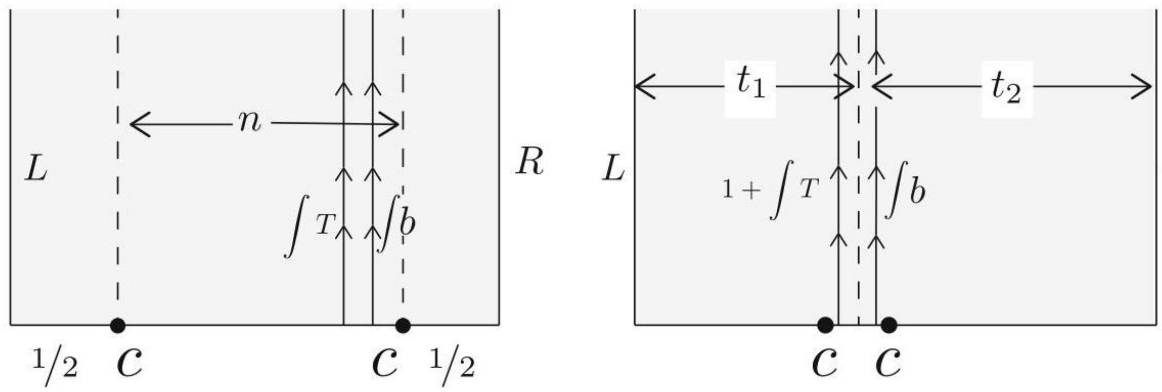

Fig. 18 Left: The n -th term in the tachyon vacuum solution by Schnabl, a wedge state of width $n+1$ with two c insertions and a B insertion. Right: The integrand in the simple tachyon vacuum solution, with left and right wedges of widths t_1 and t_2 , respectively, and insertions in between them

图 18 左: 施纳布尔快子真空解的第 n 项，是宽度为 $n+1$ 的楔形态，包含两次 c 插入和一次 B 插入。右: 简单快子真空解的被积函数，左右分别是宽度为 t_1 和 t_2 的楔形态，插入项位于两个楔形态之间

2. $F(\mathcal{K}) = \frac{1}{\sqrt{1+\mathcal{K}}}$.

Written by Erler and Schnabl, this is the simplest known regular solution, and it seems to have no subtlety [177]. It is given by

该解由埃勒和施纳布尔提出，是目前已知最简单的正则解，且似乎不存在上述微妙问题 [177]，形式为

$$\Psi = \frac{1}{\sqrt{1+\mathcal{K}}} c\,(1+\mathcal{K})\,Bc \frac{1}{\sqrt{1+\mathcal{K}}}. \tag{566}$$

Recalling that $\Omega_t = e^{-t\mathcal{K}}$, we have the representation

回顾 $\Omega_t = e^{-t\mathcal{K}}$ ，我们得到表示

$$\frac{1}{\sqrt{1+\mathcal{K}}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dt}{\sqrt{t}} e^{-t} \Omega_t \tag{567}$$

and this can be used to rewrite the solution as

由此可将解改写为

$$\Psi = \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{dt_1}{\sqrt{t_1}} \frac{dt_2}{\sqrt{t_2}} e^{-(t_1+t_2)} \Omega_{t_1} c\,(1+\mathcal{K})\,Bc\Omega_{t_2}. \tag{568}$$

We see that this time, we have a continuous sum over wedge states that goes from the identity to the sliver. The solution has a width t_1 wedge, with insertions on its right side (two ghosts sandwiching B and $(1+\mathcal{K})$ operators) glued to a width t_2 wedge (Fig. 18, right). Verifying that this solution has the right energy is a relatively short calculation using tools of conformal field theory.

可以看到这一次我们得到了从单位态到银态的楔形态连续和。该解是宽度为 t_1 的楔形态，其右侧有插入项 (夹着 B 和 $(1+\mathcal{K})$ 算符的两个鬼算符)，拼接在宽度为 t_2 的楔形态上 (图 18 右)。利用共形场论工具，验证该解能量正确是一项相当简短的计算。

Similar analytical solutions in the open superstring field theory discussed in section ”Berkovits Open Superstring Field Theory in the NS Sector” were found in [188]. References [189-191] use similar techniques for describing solutions associated with marginal deformations of the boundary conformal field theory. Some of these numerical and analytical methods have also been extended to a discussion of time dependent solutions in open string field theory [192, 193], but we shall not discuss them in this review.

在“NS 区的贝尔科维茨开超弦场论”一节讨论的开超弦场论中，类似的解析解已在文献 [188] 中找到。文献 [189-191] 使用类似技术描述边界共形场论边缘形变对应的解。部分数值和解析方法还被推广到讨论开弦场论中的含时解 [192, 193]，我们不在本综述中讨论这些内容。

An important question in this area is: Given a particular solution in open string field theory, what is its physical interpretation? For this purpose, one can use gauge-invariant observables of open string field theory. The idea for such observables originated from an open-closed vertex arising in the factorization of open string amplitudes in the associative open SFT [194, 195]. In this vertex, the open string is viewed as a semi-infinite strip, with the two halves of the final open string glued together and the closed string state inserted on the conical singularity at the string midpoint (Fig. 19).

该领域一个重要的问题是: 给定开弦场论的一个特定解, 它的物理解释是什么? 为此我们可以使用开弦场论的规范不变可观测量。这类可观测量的思想源自结合开弦场论开弦振幅因子化产生的开-闭顶点 [194, 195]。在这个顶点中，开弦被视为半无限长条带，末态开弦的两半粘接在一起，闭弦态插入在弦中点的锥奇点处 (图 19)。

Because of the conical singularity, only on-shell closed string states are naturally inserted at this singular open string midpoint, and it was shown in [8] that if the associative OSFT is supplemented by just this interaction, the Feynman rules of the theory provide a cover of all the moduli spaces of surfaces with boundaries and both open and closed string punctures. This covering property suggested the open-string gauge invariance of the open-closed vertex coupling to on-shell closed strings. Such property was noted independently by Hashimoto and Itzhaki [196] and Gaiotto et.al. [197], thus providing an observable of the open string field theory.

由于存在锥形奇点，只有在壳闭弦态能够自然地插入在这个奇异的开弦中点，文献 [8] 表明: 若仅给结合性开弦场论 (OSFT) 补充这一相互作用，该理论的费曼规则就可以覆盖所有带边界、同时带有开弦孔与闭弦孔的曲面的模空间。这种覆盖性质提示，耦合在壳闭弦的开-闭顶点满足开弦规范不变性。桥本 (Hashimoto) 与伊扎基 (Itzhaki)[196]、加约托 (Gaiotto) 等人 [197] 分别独立发现了这一性质，由此得到了开弦场论的一个可观测量。

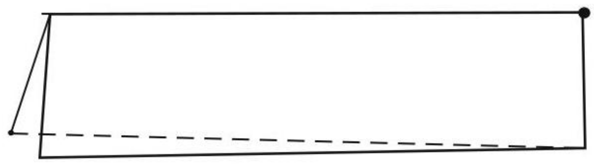

Fig. 19 The open-closed vertex used for gauge-invariant observables in OSFT. The open string state is inserted at the infinitely far past of the open string. The closed string state is inserted at the midpoint of the final open string (heavy dot), which has its two halves glued

图 19 OSFT 中用于构造规范不变可观测量的开-闭顶点。开弦态插入在开弦的无限远过去处。闭弦态插入在最终开弦的中点 (粗黑点)，开弦的两个半段在此粘合

A key step was made by Ellwood [198] who conjectured and gave evidence that the above gauge invariant observable (now called Ellwood invariant) is related to the closed string boundary state $|B\rangle$. Recall that the boundary state (for bosonic strings) is a ghost-number-three closed string state reproducing the one-point functions of arbitrary closed string operators on a disk as follows: $\langle c_0^- \mathcal{O}\rangle_{\text{disk}} = \langle B\,|c_0^-|\,\mathcal{O}\rangle$. Let $\Phi = 0$ represent the open string background used to formulate the OSFT, and $\widehat{\Phi}$ be the open string field classical solution that represents a different open string background. Moreover, let $|B_0\rangle$ and $|B_{\widehat{\Phi}}\rangle$ denote the boundary state for the original $\Phi = 0$ background and the boundary state for the background represented by $\widehat{\Phi}$, respectively. With $\langle E\,[\mathcal{O}_c]|$ denoting the ghost number two open string state such that $\langle E\,[\mathcal{O}_c]\mid \phi_o\rangle$ denotes the disk two-point function of any open string state ϕ_o and on-shell closed string state $\mathcal{O}_c$ ($Q_c\mathcal{O}_c = 0$) in the open-closed vertex, we have Ellwood's relation:

埃尔伍德 (Ellwood)[198] 取得了关键进展: 他提出猜想并给出证据，指出上述规范不变可观测量 (现称埃尔伍德不变量) 与闭弦边界态 $|B\rangle$ 相关。回顾一下，(玻色弦的) 边界态是鬼数为 3 的闭弦态，它可以再生任意闭弦算符在圆盘上的单点函数，形式如下: $\langle c_0^- \mathcal{O}\rangle_{\text{disk}} = \langle B | c_0^- | \mathcal{O}\rangle$。设 $\Phi = 0$ 为表述 OSFT 所用的开弦背景，$\widehat{\Phi}$ 是代表另一不同开弦背景的开弦场经典解。另外，分别设 $|B_0\rangle$ 和 $|B_{\widehat{\Phi}}\rangle$ 为原背景 $\Phi = 0$ 对应的边界态，以及 $\widehat{\Phi}$ 代表的背景对应的边界态。再设 $\langle E[\mathcal{O}_c]|$ 为鬼数 2 的开弦态，满足 $\langle E[\mathcal{O}_c] \mid \phi_o\rangle$ 代表任意开弦态 ϕ_o 与在壳闭弦态 $\mathcal{O}_c$ ($Q_c\mathcal{O}_c = 0$) 在开-闭顶点中的圆盘两点函数，我们就得到埃尔伍德关系:

$$-4\pi i \left\langle E[\mathcal{O}_c] \mid \widehat{\Phi} \right\rangle = \langle \mathcal{O}_c | c_0^- | B_{\widehat{\Phi}} \rangle - \langle \mathcal{O}_c | c_0^- | B_0 \rangle . \tag{569}$$

The $\mathcal{O}_c$ -based Ellwood invariant on the left-hand side computes the difference of one-point functions of $\mathcal{O}_c$ in the final and original open string backgrounds. For additional details on normalization of the various quantities in this relation, see, for example, [14]. An extension of Ellwood's construction to deal with general closed string states and thus get all the information of the boundary state was given in [199]. An explicit procedure for constructing the boundary state corresponding to a given solution in open string field theory was given in [200].

左侧基于 $\mathcal{O}_c$ 的埃尔伍德不变量，计算了最终开弦背景与原开弦背景中 $\mathcal{O}_c$ 单点函数的差值。关于该关系中各物理量归一化的更多细节，例如可参见文献 [14]。文献 [199] 推广了埃尔伍德的构造，使其可处理一般闭弦态，从而得到边界态的全部信息。文献 [200] 给出了开弦场论中，构造给定解对应边界态的显式步骤。

Mass Renormalization and Vacuum Shift

质量重整化与真空平移

In this section, we shall first briefly review where the world-sheet formalism breaks down in dealing with mass renormalization and vacuum shift and then describe how string field theory can be used to resolve these problems.

在本节中，我们将首先简要回顾世界面形式论处理质量重整化和真空平移时在何处失效，随后说明弦场论如何能够用于解决这些问题。

Issue with Mass Renormalization

质量重整化问题

It is commonly believed that the world-sheet formalism for string theory allows us to calculate on-shell amplitudes, i.e., S-matrix elements. This is not strictly true, however. In world-sheet formalism, we define on-shell states as appropriate representatives of BRST cohomology, normalized according to certain rules, and then compute the S-matrix element as integrals of correlation function of the corresponding vertex operators on the Riemann surfaces. If we translate this to the language of string field theory, it will amount to defining on-shell states as appropriately normalized solutions to the linearized classical equations of motion and computing the S-matrix element as the convolution of these wave-functions with the off-shell Green's functions. However, we know from our study of quantum field theory that except in very special circumstances, this is not the correct procedure for computing the S-matrix. There are two ways this procedure fails:

> 人们普遍认为弦论的世界面形式可以计算在壳振幅，也就是 S 矩阵元。但这并不完全正确。在世界面形式中，我们将在壳态定义为 BRST 上同调的合适代表，按照特定规则归一化，再将 S 矩阵元计算为对应顶点算符在黎曼曲面上的关联函数积分。如果将其转换到弦场论的语言，这相当于把在壳态定义为线性化经典运动方程的合适归一化解，将 S 矩阵元计算为这些波函数与离壳格林函数的卷积。然而我们从量子场论研究中得知，除了非常特殊的情况，这套计算 S 矩阵的流程并不正确。该流程的问题分为两点:

1. We know that in quantum field theory, the masses of various fields can get renormalized. Therefore, a state that is on-shell at tree level is no longer on-shell at the loop level. If we ignore this effect and try to compute the convolution of the tree-level wave-function of the state with the Green's function, we shall encounter divergences. In perturbation theory, these divergences show up as self-energy diagrams on external legs. If we try to evaluate them by setting the external momenta to satisfy the tree-level on-shell condition, then we shall get divergent results since these diagrams will contain on-shell internal propagators. An example of this has been shown in Fig. 20a where the internal line carrying momentum k is forced to be on-shell if we require the external state to be on-shell.

> 1. 我们知道，在量子场论中，各类场的质量都会发生重整化。因此，树图层次的在壳态在圈图层次不再是在壳态。如果我们忽略这一效应，直接用该态的树图波函数和格林函数做卷积计算，就会出现发散。在微扰论中，这些发散表现为外腿的自能图。如果我们强行让外动量满足树图的在壳条件来计算这些图，就会得到发散结果，因为这些图中包含在壳内传播子。图 20a 给出了一个例子: 如果我们要求外态在壳，那么携带动量 k 的内线就必须是在壳的。

2. Even in special cases where the mass renormalization effect is absent, e.g., for massless gauge particles, Goldstone bosons, etc., there may be wave-function renormalization effects that tell us that a correctly normalized state for computing tree-level S-matrix elements may not be correctly normalized at the loop level. If we do not take this into account, we do not get a unitary S-matrix.

> 2. 即使在不存在质量重整化效应的特殊情况，比如无质量规范粒子、戈德斯通玻色子等，仍然可能存在波函数重整化效应: 适用于树图 S 矩阵元计算的正确归一化态，在圈图层次的归一化不一定正确。如果不考虑这一点，我们就无法得到幺正的 S 矩阵。

We encounter similar issues in the naive world-sheet approach to the computation of the string S-matrix. For example, if we take the vertex operators to be elements of the BRST cohomology satisfying the tree-level on-shell condition and try to evaluate a one loop amplitude, we shall get divergent results from degenerate Riemann surfaces of the kind shown in Fig. 20b. The relation to the diagram in Fig. 20a follows from the Schwinger parameter representation of the Siegel gauge propagator that is needed to express string theory Feynman diagrams as world-sheet contribution:

在弦 S 矩阵计算的朴素世界面方法中，我们也会遇到类似问题。例如，如果我们取顶点算符为满足树图在壳条件的 BRST 上同调元素，尝试计算单圈振幅，就会从图 20b 所示的这类退化黎曼曲面得到发散结果。它和图 20a 中 diagrams 的关联来自 Siegel 规范传播子的施温格参数表示——将弦论费曼图表示为世界面贡献时需要用到这个表示:

$$\frac{1}{L_0+\bar{L}_0}\delta_{L_0,\bar{L}_0}=\frac{1}{2\pi}\int_{|q|\leq 1}\frac{d^2q}{|q|^2}q^{L_0}\bar{q}^{\bar{L}_0}. \tag{570}$$

For $L_0 > 0$, this is an identity, but for $L_0 < 0$, the right-hand side diverges from the $q \simeq 0$ region even though the left-hand side is finite. In this case, the left-hand side can be regarded as the analytic continuation of the right hand side. For $L_0 = 0$, however, both sides are divergent. On the left-hand side, such divergences occur from the on-shell internal propagator of string field theory as in the case of Fig. 20a. On the right hand side, such divergences, coming from the $q \simeq 0$ region, are associated with degenerate Riemann surfaces of the kind shown in Fig. 20b once we identify q with the plumbing fixture variable.

对于 $L_0 > 0$ ，这是一个恒等式；但对于 $L_0 < 0$ ，即使左侧是有限的，右侧也会从 $q \simeq 0$ 区域产生发散，此时左侧可以看作是右侧的解析延拓。但对于 $L_0 = 0$ ，两侧都是发散的。在左侧，这类发散和图 20a 的情况一样，来自弦场论的在壳内传播子；在右侧，当我们把 q 识别为 plumbing fixture 变量后，来自 $q \simeq 0$ 区域的这类发散就对应图 20b 所示的这类退化黎曼曲面。

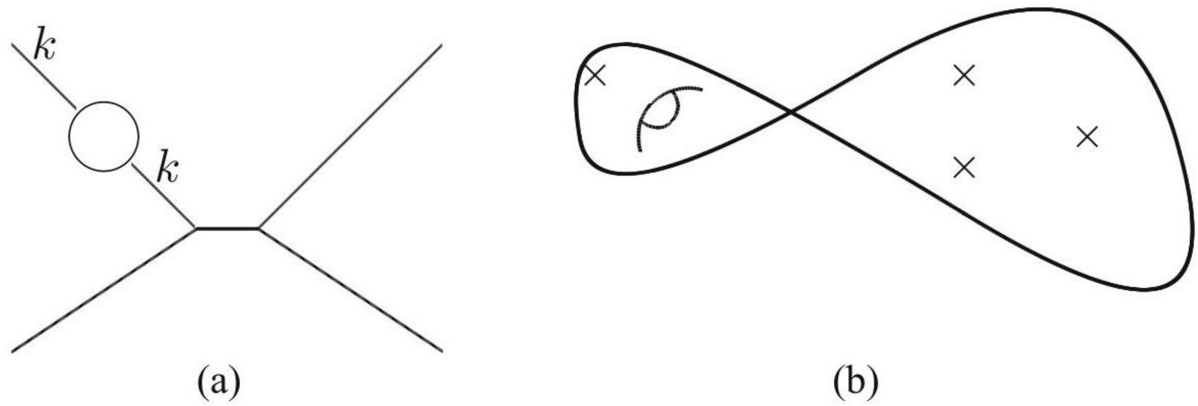

Fig. 20 (a) Mass renormalization diagram in quantum field theory. (b) Divergent contribution to string theory amplitudes associated with one loop mass renormalization. The crosses denote vertex operators of external states

图 20 (a) 量子场论中的质量重整化图。(b) 单圈质量重整化给弦论振幅带来的发散贡献。叉号代表外态的顶点算符

In the world-sheet theory, there is no systematic procedure for dealing with such divergences, [28] but in string field theory, we can use the usual insights from quantum field theory. In particular, the LSZ prescription tells us that instead of dealing with diagrams of the type shown in Fig. 20a in isolation, we need to first compute the off-shell propagator by summing over all self-energy diagrams, locate the poles of the propagator to compute renormalized masses and the residues at the poles to compute the wave-function renormalizations, and then evaluate the amputated Green's function at the (loop corrected) on-shell external momenta, renormalized by the wave-function renormalization factors. Use of amputated Green's function ensures that there are no self-energy insertions on external legs as in Fig. 20a, and we get a finite result.

在世界面理论中，没有处理这类发散的系统方法，[28] 但在弦场论中，我们可以沿用量子场论的成熟思路。特别是根据 LSZ 公式，我们不需要单独处理图 20a 这类图，而是可以先对所有自能图求和计算离壳传播子，找到传播子的极点来计算重整化质量，通过极点留数计算波函数重整化，再将截肢格林函数在经圈修正的外在壳动量处求值，再用波函数重整化因子做重整化。使用截肢格林函数可以保证外腿不会出现图 20a 那样的自能插入，最终得到有限结果。

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[28] When mass renormalization is absent and hence there are no divergences, the amplitude in the world-sheet theory may be still ambiguous since the integrals become only conditionally convergent [201]. Working within the conventional world-sheet formalism, [201] showed that these are associated with possible ambiguities in the finite part of the wave-function renormalization. We need to ensure, however, that when the finite part of the wave-function renormalization is fixed in one amplitude, we use the same renormalization prescription for all other amplitudes. To do this, one needs string field theory; in the conventional world-sheet description, one cannot relate the regulator in one amplitude to a regulator in a different amplitude. A similar remark holds for the situation where tadpoles of massless moduli fields vanish, but there is an ambiguity related to redefinition of the moduli fields.

[28] 当不存在质量重整化因而没有发散时，世界面理论中的振幅仍可能存在歧义，因为相关积分仅是条件收敛的 [201]。在传统世界面形式框架下，文献 [201] 指出这些歧义与波函数重整化有限部分中可能存在的不确定性相关。但我们需要保证，当一个振幅中波函数重整化的有限部分被固定后，我们对所有其他振幅都采用相同的重整化方案。要做到这一点就需要弦场理论；在传统世界面描述中，我们无法将一个振幅中的正则化项与另一个不同振幅中的正则化项联系起来。类似的结论也适用于以下情况: 零质量模场的蝌蚪图消失，但存在模场重定义带来的歧义。

Since string field theory can be regarded as a regular quantum field theory, we can follow the same procedure to get finite string amplitudes. An extra complication arises because string theory is a gauge theory; not all the poles in the gauge-fixed propagator are physical, and we have to carefully sort out the physical poles from the others. This can be done [202, 203], but we shall later describe a way to get around this using the 1PI effective action [77, 204].

由于弦场理论可以被看作常规量子场论，我们可以遵循相同的流程得到有限的弦振幅。弦理论是规范理论，因此会带来额外的复杂性: 规范固定后的传播子中并非所有极点都是物理的，我们必须仔细将物理极点与其他极点区分开。这一点是可以做到的 [202, 203]，不过我们之后会介绍一种利用 1PI 有效作用量绕开该问题的方法 [77, 204]。

Issues with Vacuum Shift

真空移位问题

Consider a massless scalar field theory with action

考虑一个无质量标量场论，其作用量为

$$S=-\int d^Dx\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{g_s}{3!}\phi^3\right]. \tag{571}$$

At one loop order, this theory develops a tadpole of the ϕ field. If we ignore its presence and try to compute amplitudes using the usual Feynman rules, we run into divergences from internal ϕ propagators that carry zero momentum. An example of this has been shown in Fig. 21a where we see an intermediate massless propagator carrying strictly zero momentum due to momentum conservation. As we discussed above, the analogous divergences arise in string theory via degenerate Riemann surfaces of the type shown in Fig. 21b.

在单圈阶，该理论会产生 ϕ 场的蝌蚪图。如果我们忽略它的存在，尝试用常规费曼规则计算振幅，就会遇到来自携带零动量的内部 ϕ 传播子的发散。图 21a 给出了一个例子，我们可以看到根据动量守恒，中间的无质量传播子携带严格为零的动量。正如我们之前讨论的，类似的发散也出现在弦论中，来自图 21b 所示的这类退化黎曼曲面。

In quantum field theory, there is a well-defined procedure for dealing with this problem. Tadpole diagrams may be regarded as a contribution to the effective action that is linear in the fields. Therefore, the effective action takes the form

在量子场论中，有一套定义明确的方法处理这个问题。蝌蚪图可以被看作是对有效作用量的贡献，它是场的线性项。因此，有效作用量的形式为

$$S=-\int d^Dx\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{g_s}{3!}\phi^3+cg_s\phi+\mathcal{O}\left(\phi^2\right)\right], \tag{572}$$

where c is some constant. Therefore, the true extrema of the effective action are at

其中 c 是某个常数。因此，有效作用量的真实极值点位于

$$\phi=\pm\sqrt{-2c}+\mathcal{O}\left(g_s\right). \tag{573}$$

If $c<0$, then there are two real solutions, one of which is a minimum, and the other one is a maximum of the effective potential. In this case, we can define perturbative amplitudes by expanding the fields around the minimum. This minimum at non-zero ϕ represents the vacuum shift from $\phi=0$. Of course, the theory is non-perturbatively ill-defined since the potential is unbounded from below, but that will not be relevant for our discussion (and can be avoided by adding higher order terms in the classical potential). If c is positive, then there is no extremum at real values of ϕ , and the theory is ill-defined, at least perturbatively.

如果 $c<0$ ，则存在两个实解: 一个是有效势的极小值，另一个是极大值。在这种情况下，我们可以通过在场空间围绕极小值展开来定义微扰振幅。非零 ϕ 处的这个极小值就代表了相对于 $\phi=0$ 的真空移位。当然，该理论从非微扰角度来看是不适定的，因为有效势没有下界，但这对我们的讨论并不重要 (而且可以通过在经典势中添加高阶项来避免这个问题)。如果 c 为正，那么在 ϕ 的实数值范围内不存在极值，该理论至少在微扰框架下是不适定的。

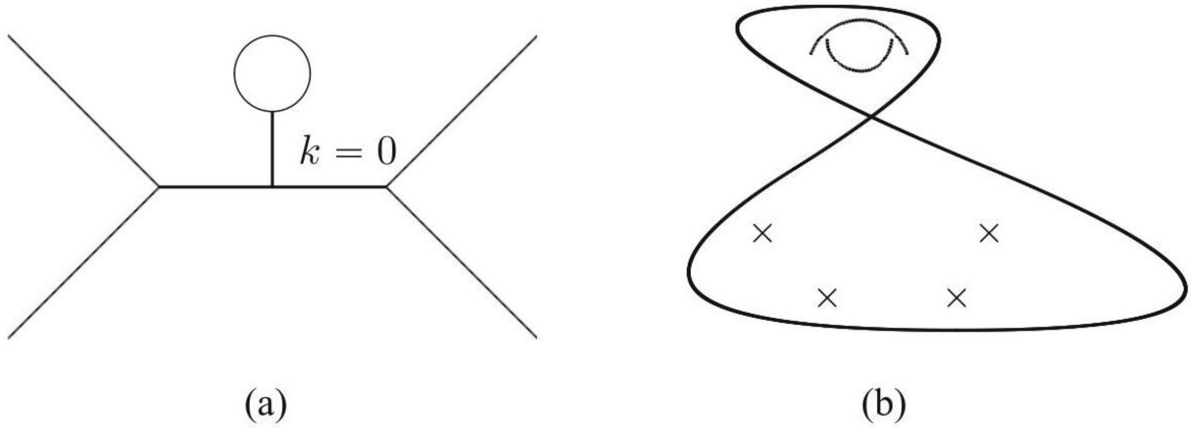

Fig. 21 (a) Massless tadpole diagram in quantum field theory. (b) Divergent contribution to string theory amplitudes associated with massless tadpoles. The crosses denote vertex operators of external states

图 21 (a) 量子场论中的无质量蝌蚪图。(b) 与无质量蝌蚪相关的弦论振幅发散贡献。十字标记外态的顶点算符

In string theory, there is no known systematic way to solve the tadpole problem in the world-sheet theory although some suggestions have been made in [205]. The main idea of [205] is similar in spirit to that in quantum field theory, i.e. cancel the loop generated tadpoles against a tadpole generated at tree level by going away from the conformal background. However, both the loop-induced tadpole and the tree-level tadpoles involve regulators, and a priori there is no systematic way of extracting an unambiguous finite answer after cancelling these divergences. In string field theory, on the other hand, we can follow the same procedure as in a regular quantum field theory and arrive at an unambiguous result. [29] We shall illustrate this below.

在弦论中，目前世界面理论中没有已知的系统方法解决蝌蚪图问题，尽管文献 [205] 已经提出了一些思路。[205] 的核心思路在精神上与量子场论类似，即通过偏离共形背景，让圈图产生的蝌蚪与树图产生的蝌蚪抵消。然而，圈诱导蝌蚪和树图蝌蚪都依赖正则化，且目前没有系统方法能在发散抵消后提取出明确的有限结果。另一方面，在弦场论中，我们可以沿用普通量子场论的相同步骤，得到明确的结果。[29] 我们将在下文举例说明。

Solution of the Vacuum Shift Problem Using String Field Theory

用弦场理论求解真空偏移问题

In the following, we shall consider the case of closed superstring theory, but the results also hold in other string theories with trivial changes.

下文我们将讨论闭超弦理论的情况，但所得结果经过微小调整也适用于其他弦理论。

Equations of motion derived from the 1PI effective action (268) take the form

从 1PI 有效作用量 (268) 导出的运动方程形式为

$$Q\left(|\Psi\rangle - \mathcal{G}|\widetilde{\Psi}\rangle\right) = 0$$

$$Q|\widetilde{\Psi}\rangle + \sum_{n=0}^{\infty} \frac{1}{n!}[\Psi^n]_{1\mathrm{PI}} = 0. \tag{574}$$

Our goal will be to look for solution to this equation in perturbation theory, with perturbation parameter the coupling constant g_s . Recall that g_s appears in the above equations through the definition of the product $[\Psi^n]$ where g_s is the expansion parameter for the sum over genus (see, e.g., (109)).

我们的目标是在微扰论中寻找该方程的解，微扰参数为耦合常数 g_s 。回顾可知， g_s 通过乘积 $[\Psi^n]$ 的定义出现在上述方程中，其中 g_s 是亏格求和的展开参数 (参见例如 (109))。

For a translationally invariant vacuum solution, we can work at zero momentum. For simplicity, we set the RR fields to zero so that we have only the NS sector fields. In this case, $\mathcal{G} = 1$, and we can set

对于平移不变的真空解，我们可以在零动量处计算。为简化，我们将 RR 场设为零，仅保留 NS 扇区场。这种情况下， $\mathcal{G} = 1$ ，因此我们可以设

$$|\widetilde{\Psi}\rangle = |\Psi\rangle, \tag{575}$$

so that the equation takes the form

使得方程变为如下形式

$$Q|\Psi\rangle + \sum_{n=0}^{\infty} \frac{1}{n!}[\Psi^n]_{1\mathrm{PI}} = 0. \tag{576}$$

---

[29] A hybrid approach that does not use the full power of string field theory, but uses enough ingredients involving off-shell amplitudes, was used in [206] to address the tadpole problem.

[29] 文献 [206] 采用了一种混合方法处理蝌蚪问题，该方法没有用到弦场理论的全部能力，但用到了足够多包含离壳振幅的要素。

We can try to solve this equation iteratively in a power series in g_s , starting with the leading order solution $|\Psi_0\rangle = 0$. We shall assume that the solution can be expanded in integer powers of g_s , but the analysis can be easily extended to the situation where the solution has fractional powers of g_s . If $|\Psi_k\rangle$ denotes the solution with all powers of g_s up to and including g_s^k , then a formal iterative solution is given by

我们可以尝试将该方程按 g_s 的幂级数迭代求解，从领头阶解 $|\Psi_0\rangle = 0$ 开始。我们假设解可以按 g_s 的整数次幂展开，但若解包含 g_s 的分数次幂，该分析也可以轻松拓展。若 $|\Psi_k\rangle$ 表示包含所有不超过 g_s^k 次 g_s 幂次的解，那么形式迭代解可写为

$$|\Psi_{k+1}\rangle = -\frac{b_0^+}{L_0^+}\sum_{n=0}^{k+1}\frac{1}{n!}\left[\Psi_k^n\right]_{1\mathrm{PI}} + \mathcal{O}\left(g_s^{k+2}\right). \tag{577}$$

Since $|\Psi_{k+1}\rangle$ must be a solution up to and including order g_s^{k+1} , terms of $\mathcal{O}\left(g_s^{k+2}\right)$ can be ignored. This means that the $\mathcal{O}\left(g_s^{k+2}\right)$ terms arising from the $\left[\Psi_k^n\right]$ products (due to the infinite sum over genus) are to be ignored. The state $|\Psi_{k+1}\rangle$ is a solution because one can easily check that the lower-order equations imply that the state $\sum_{n=0}^{k+1}\frac{1}{n!}\left[\Psi_k^n\right]_{1\mathrm{PI}}$ is BRST invariant up to corrections of order g_s^{k+2} .

由于 $|\Psi_{k+1}\rangle$ 必须是包含到 g_s^{k+1} 阶的解，因此 $\mathcal{O}\left(g_s^{k+2}\right)$ 项可以忽略。这意味着我们可以忽略来自 (亏格无穷求和导致的) $\left[\Psi_k^n\right]$ 乘积的 $\mathcal{O}\left(g_s^{k+2}\right)$ 项。态 $|\Psi_{k+1}\rangle$ 是一个解，因为不难验证，低阶方程意味着态 $\sum_{n=0}^{k+1}\frac{1}{n!}\left[\Psi_k^n\right]_{1\mathrm{PI}}$ 在 g_s^{k+2} 阶修正范围内是 BRST 不变的。

The problem with this formal solution, however, is that $\sum_{n=0}^{k+1}\frac{1}{n!}\left[\Psi_k^n\right]_{1\mathrm{PI}}$ may have components along states with $L_0^+ = 0$, and hence $1/L_0^+$ acting on these states is ill-defined. To address this, we introduce a projection operator $\mathbf{P}$ into the $L_0^+ = 0$ states and write

然而，该形式解存在一个问题: $\sum_{n=0}^{k+1}\frac{1}{n!}\left[\Psi_k^n\right]_{1\mathrm{PI}}$ 可能存在沿 $L_0^+ = 0$ 态的分量，因此作用在这些态上的 $1/L_0^+$ 是不良定义的。为解决该问题，我们向 $L_0^+ = 0$ 态引入投影算符 $\mathbf{P}$ 并写作

$$|\Psi_{k+1}\rangle = -\frac{b_0^+}{L_0^+}(1-\mathbf{P})\sum_{n=0}^{k+1}\frac{1}{n!}\left[\Psi_k^n\right]_{1\mathrm{PI}} + |\psi_{k+1}\rangle + \mathcal{O}\left(g_s^{k+2}\right), \mathbf{P}\,|\psi_{k+1}\rangle = |\psi_{k+1}\rangle,$$

$$Q\,|\psi_{k+1}\rangle = -\mathbf{P}\sum_{n=0}^{k+1}\frac{1}{n!}\left[\Psi_k^n\right]_{1\mathrm{PI}} + \mathcal{O}\left(g_s^{k+2}\right).$$

(578)

The first line is an ansatz for $|\Psi_{k+1}\rangle$, which introduces an unknown string field $|\psi_{k+1}\rangle$ with components along the subspace of states of vanishing L_0^+ and defined up to irrelevant $\mathcal{O}\left(g_s^{k+2}\right)$ terms. The equation on the second line is the key one. If it can be solved for $|\psi_{k+1}\rangle$, then one can show that the equation of motion (576) is satisfied to $\mathcal{O}\left(g_s^{k+1}\right)$. This can be checked by applying $\mathbf{P}$ and $(1-\mathbf{P})$ on (576).

第一行是 $|\Psi_{k+1}\rangle$ 的一个拟设，它引入了一个未知弦场 $|\psi_{k+1}\rangle$ ，该弦场在消失 L_0^+ 的态子空间上存在分量，且定义到不相关的 $\mathcal{O}\left(g_s^{k+2}\right)$ 项。第二行的方程是核心方程。如果能对 $|\psi_{k+1}\rangle$ 求解，就能证明运动方程 (576) 在 $\mathcal{O}\left(g_s^{k+1}\right)$ 阶成立，这可以通过将 $\mathbf{P}$ 和 $(1-\mathbf{P})$ 作用于 (576) 来验证。

There is a possible obstruction to solving for $|\psi_{k+1}\rangle$. As already mentioned, the right-hand side of this equation is BRST invariant to order g_s^{k+1} as long as $|\Psi_k\rangle$ satisfies the equation of motion to order g_s^k . Therefore, if the right-hand side is BRST trivial, i.e., has the form $Q\,|s_{k+1}\rangle$ for some state $|s_{k+1}\rangle$, we can simply set $|\psi_{k+1}\rangle = |s_{k+1}\rangle$. However, if the right-hand side contains non-trivial elements of the BRST cohomology, then this equation cannot be solved, and there is a genuine obstruction to finding a perturbative solution. Another feature of the equation for $|\psi_{k+1}\rangle$ is that when the solution exists, it is not unique since we can add any linear combination of BRST invariant states with coefficients of order g_s^{k+1} to $|\psi_{k+1}\rangle$, and the equation is still satisfied since these terms are killed by Q .

求解 $|\psi_{k+1}\rangle$ 可能存在阻碍。如前所述，只要 $|\Psi_k\rangle$ 满足 g_s^k 阶的运动方程，该方程的右手边就是 g_s^{k+1} 阶 BRST 不变的。因此，如果右手边是 BRST 平凡的，即对某个态 $|s_{k+1}\rangle$ 具有形式 $Q\,|s_{k+1}\rangle$，我们可以直接设 $|\psi_{k+1}\rangle = |s_{k+1}\rangle$。但若右手边包含 BRST 上同调的非平凡元素，该方程就无解，寻找微扰解会存在真实阻碍。$|\psi_{k+1}\rangle$ 满足的方程还有一个性质: 当解存在时，解不唯一，因为我们可以给 $|\psi_{k+1}\rangle$ 加上任意 BRST 不变态的线性组合，其系数为 g_s^{k+1} 阶，方程仍然成立，因为这些项会被 Q 湮灭。

Both these features of the solution have analogs in a conventional quantum field theory whose classical potential has flat directions. Let ϕ denote such a flat direction. If at some order quantum corrections generate a non-zero term on the right-hand side of the equation of motion of ϕ , then equations of motion cannot be solved by an iterative procedure since the equation of motion of ϕ will have a vanishing left-hand side but a non-vanishing right-hand side. On the other hand if quantum corrections do not generate a non-zero term on the right-hand side of the equation of motion of ϕ to a given order (say g_s^k), then at that order, the solution will be ambiguous since given a solution, we can always shift ϕ by a term of order g_s^k and still have a solution.

解的这两个性质在经典势具有平坦方向的传统量子场论中都有对应。设 ϕ 是这样一个平坦方向。如果在某一阶量子修正产生了非零项出现在 ϕ 运动方程的右手边，运动方程就无法通过迭代法求解，因为 ϕ 的运动方程左手边为零但右手边非零。反之，如果量子修正没有在给定阶 (比如 g_s^k) 的 ϕ 运动方程右手边产生非零项，那么在该阶解就存在歧义，因为给定一个解后，我们总可以给 ϕ 加上一个 g_s^k 阶的项，结果仍然是解。

Some time we may be able to use the ambiguity in determining the solution at a given order to fix the problem of lifting of flat directions at a higher order. Indeed, even though we can add to $|\psi_{k+1}\rangle$ any linear combination of BRST-invariant states with coefficients of order g_s^{k+1} , at higher order, these coefficients may be fixed by the condition that the right-hand side of the equation for $|\psi_{k'}\rangle$ at higher order does not have non-trivial elements of the BRST cohomology. An illustration of the procedure outlined here can be found in [207]. This again has its counterpart in quantum field theory where quantum corrections may fix the location of the solution along a flat direction instead of destroying the solution.

有时我们可以利用给定阶求解解的歧义，解决更高阶平坦方向提升的问题。事实上，即使我们可以给 $|\psi_{k+1}\rangle$ 加上任意 BRST 不变态的线性组合，系数为 g_s^{k+1} 阶，在更高阶，这些系数可以通过“更高阶 $|\psi_{k'}\rangle$ 方程的右手边不包含 BRST 上同调非平凡元素”这一条件固定。该流程的示例可见文献 [207]。这在量子场论中也有对应: 量子修正会固定解沿平坦方向的位置，而非破坏解的存在性。

This procedure can be generalized to analyze the shift of RR background as well. Examples of this can be found in [208, 209] .

这个流程可以推广到分析 RR 背景移动，相关示例可见 [208, 209] 。

Solution of the Mass Renormalization Problem Using String Field Theory

用弦场理论求解质量重整化问题

Consider a vacuum solution $|\Psi_v\rangle$ to the equations of motion, as discussed in the previous subsection, with the assumption that the string field is in the NS sector, and thus we have $|\widetilde{\Psi}\rangle = |\Psi\rangle = |\Psi_v\rangle$. We can study small fluctuations around the vacuum solution to find the renormalized masses. For this we express the string field as

考虑上一小节讨论的运动方程的真空解 $|\Psi_v\rangle$,假设弦场处于 NS 扇区,因此我们得到 $|\widetilde{\Psi}\rangle = |\Psi\rangle = |\Psi_v\rangle$ 。我们可以研究真空解附近的小微扰，以得到重整化质量。为此我们将弦场表示为

$$|\Psi\rangle = |\Psi_v\rangle + |\Phi\rangle, \; |\widetilde{\Psi}\rangle = |\Psi_v\rangle + |\widetilde{\Phi}\rangle. \tag{579}$$

The fluctuations in these two fields are not identical because the result $|\widetilde{\Psi}\rangle = |\Psi\rangle$ holds only for the vacuum solution. We work to linear order in the fluctuations $|\Phi\rangle$, $|\widetilde{\Phi}\rangle$. It is easy to check that to linear order, equations of motion for the fluctuating fields are

这两个场的涨落并不等价，因为结果 $|\widetilde{\Psi}\rangle = |\Psi\rangle$ 仅对真空解成立。我们将讨论限定在涨落 $|\Phi\rangle$ 、$|\widetilde{\Phi}\rangle$ 的线性阶。不难验证，在线性阶下，涨落场满足的运动方程为

$$Q\left(|\Phi\rangle - \mathcal{G}|\widetilde{\Phi}\rangle\right) = 0, \; Q|\widetilde{\Phi}\rangle + K|\Phi\rangle = 0, \tag{580}$$

where K is an operator acting on string fields as follows:

其中 K 是作用在弦场上的算符，定义如下:

$$K|A\rangle \equiv \sum_{n=0}^{\infty} \frac{1}{n!} [\Psi_v^n A] \tag{581}$$

for any state $|A\rangle$. Since Q commutes with $\mathcal{G}$, the two equations can be combined to give

对任意态 $|A\rangle$ 成立。由于 Q 与 $\mathcal{G}$ 对易，我们可以将两个方程合并得到

$$(Q + \mathcal{G}K)|\Phi\rangle = 0. \tag{582}$$

This is a linear equation for the fluctuation string field. Moreover, using the equations of motion (576) for Ψ_v , the operator K can be shown to satisfy the identity

这是涨落弦场满足的线性方程。此外，利用 Ψ_v 满足的运动方程 (576)，可以证明算符 K 满足等式

$$QK + KQ + K\mathcal{G}K = 0. \tag{583}$$

Our goal will be to find solutions to (582) in a series expansion in g_s , starting with a leading order solution representing a BRST invariant state. To do this systematically, we note that at tree level, the elements of BRST cohomology occur at $L_0^+ = 0$. The operator L_0^+ , acting on a state with momentum k , is given by

我们的目标是对 g_s 按级数展开求解 (582)，从表示 BRST 不变态的领头阶解开始。为了系统完成这一步，我们注意到树图阶，BRST 上同调的元素出现在 $L_0^+ = 0$ 处。作用在动量为 k 的态上的算符 L_0^+ 由下式给出

$$2L_0^+ = \left(k^2 + \widehat{M}^2\right), \tag{584}$$

where the operator $\widehat{M}^2$ represents the oscillator contribution to the operator in the noncompact theory, while in a compactified theory, it also includes contributions from the CFT associated with the compact directions. $\widehat{M}^2$ has discrete spectrum of degenerate eigenvalues. We shall focus on a set of degenerate eigenstates carrying a particular eigenvalue m^2 of $\widehat{M}^2$. For states of momentum k and $\widehat{M}^2$ eigenvalue m^2 , the $L_0^+ = 0$ condition now takes the form $k^2 + m^2 = 0$.

其中算符 $\widehat{M}^2$ 代表非紧致理论中该算符来自谐振子的贡献，而在紧致化理论中，它还包含紧致方向关联共形场论的贡献。$\widehat{M}^2$ 具有离散的简并本征值谱。我们将聚焦于一组携带 $\widehat{M}^2$ 特定本征值 m^2 的简并本征态。对于动量为 k 且 $\widehat{M}^2$ 本征值为 m^2 的态，$L_0^+ = 0$ 条件现在形如 $k^2 + m^2 = 0$ 。

We expect that quantum-corrected equations of motion (582) will have a solution at $k^2 + m^2 + \mathcal{O}(g_s) = 0$. We consider states carrying a fixed momentum k satisfying $k^2 + m^2 \simeq \mathcal{O}(g_s)$, and define a projection operator P that projects onto states with $\widehat{M}^2 = m^2$. For any such fixed k , this is a finite dimensional vector space V_m .

我们预期量子修正后的运动方程 (582) 在 $k^2 + m^2 + \mathcal{O}(g_s) = 0$ 处存在解。我们考虑携带固定动量 k 且满足 $k^2 + m^2 \simeq \mathcal{O}(g_s)$ 的态，定义投影算符 P 将态投影到满足 $\widehat{M}^2 = m^2$ 的态上。对任意这样的固定 k ，这是一个有限维向量空间 V_m 。

The algorithm to produce a solution to (582) begins by picking an order we want to work to. Assume that we want to work to order g_s^n for some $n \geq 1$. A solution $|\Phi_n\rangle$ to order g_s^n is constructed iteratively as follows:

求解 (582) 的算法首先需要选定我们的计算阶数。假设我们要计算到 g_s^n 阶，其中 $n \geq 1$ 任意。我们可以按如下迭代构造到 g_s^n 阶的解 $|\Phi_n\rangle$:

$$|\Phi_0\rangle = |\phi_n\rangle, \ |\Phi_{\ell+1}\rangle = -\frac{b_0^+}{L_0^+}(1-P)\,\mathcal{G}K\,|\Phi_\ell\rangle + |\phi_n\rangle, \tag{585}$$

with $|\phi_n\rangle$ satisfying

其中 $|\phi_n\rangle$ 满足

$$P\,|\phi_n\rangle = |\phi_n\rangle, \ Q\,|\phi_n\rangle = -P\mathcal{G}K\,|\Phi_{n-1}\rangle + \mathcal{O}\left(g_s^{n+1}\right). \tag{586}$$

The procedure to solve these equations is as follows:

求解这些方程的步骤如下:

1. We begin by taking $|\phi_n\rangle$ to be an arbitrary vector in V_m , labelled by a finite number of parameters.

1. 首先取 $|\phi_n\rangle$ 为 V_m 中的任意矢量，由有限个参数标记。

2. We then solve (585) for $|\Phi_\ell\rangle$ iteratively, keeping terms up to order g_s^n in the expression for K . Since $k^2 + m^2 = \mathcal{O}(g_s)$, and the projection operator $(1-P)$ removes states with $\widehat{M}^2 = m^2$, the $1/L_0^+$ acting on the state on the right-hand side gives finite result without any inverse power of g_s . Repeating this process n -times, we arrive at $|\Phi_n\rangle$. However, the result depends linearly on the particular vector $|\phi_n\rangle$ that we have chosen in the subspace projected by P .

2. 随后我们对 $|\Phi_\ell\rangle$ 迭代求解方程 (585), 在 K 的表达式中保留最高到 g_s^n 阶的项。由于 $k^2+m^2 = \mathcal{O}(g_s)$ ，且投影算符 $(1-P)$ 会移除带 $\widehat{M}^2 = m^2$ 的态，因此作用于右侧态的 $1/L_0^+$ 会给出有限结果，不包含 g_s 的任何负幂次。将该过程重复 n 次后，我们得到 $|\Phi_n\rangle$ 。但结果线性依赖于我们在 P 投影的子空间中选取的特定向量 $|\phi_n\rangle$ 。

3. We now substitute the expression for Φ_{n-1} computed in the previous steps into (586). This gives a linear equation for $|\phi_n\rangle$. Since $|\phi_n\rangle$ takes value in finite dimensional Hilbert space, these equations can be solved explicitly. Solutions that exist for generic momentum k can be identified as pure gauge states and be discarded. However, there will be additional solutions for special values of k^2 close to $-m^2$. These represent physical states, and the corresponding values of $-k^2$ give the physical mass [2] of the states.

3. 现在我们将前一步计算得到的 Φ_{n-1} 表达式代入 (586)，得到关于 $|\phi_n\rangle$ 的线性方程。由于 $|\phi_n\rangle$ 取值于有限维希尔伯特空间，该方程组可以显式求解。对任意动量子 k 都存在的解可被识别为纯规范态并丢弃。但在靠近 $-m^2$ 的特殊 k^2 取值处会存在额外解，这些解代表物理态，对应的 $-k^2$ 取值给出该态的物理质量 [2] 。

To show that this procedure generates a solution to (580) to order g_s^n , we can proceed as follows. First of all, using (585),(586), and the fact that K is of order g_s , one can prove iteratively in ℓ that $(Q+\mathcal{G}K)|\Phi_\ell\rangle$ vanishes to order $g_s^{\ell+1}$, $(1-P)Q\mathcal{G}K|\Phi_\ell\rangle$ vanishes to order $g_s^{\ell+2}$, and $|\Phi_\ell\rangle$ and $|\Phi_{\ell-1}\rangle$ differ by term of order g_s^ℓ . This gives

为了说明该过程能得到 (580) 精确到 g_s^n 阶的解，我们可以按如下步骤推导: 首先，利用 (585)、(586) 以及 K 是 g_s 阶这一性质，可以对 ℓ 做迭代证明: $(Q+\mathcal{G}K)|\Phi_\ell\rangle$ 精确到 $g_s^{\ell+1}$, $(1-P)Q\mathcal{G}K|\Phi_\ell\rangle$ 阶为零，$g_s^{\ell+2}$ 精确到 $g_s^{\ell+2}$ 阶为零，且 $|\Phi_\ell\rangle$ 与 $|\Phi_{\ell-1}\rangle$ 仅相差 g_s^ℓ 阶项。由此可得

$$(Q+\mathcal{G}K)|\Phi_n\rangle = \mathcal{O}\left(g_s^{n+1}\right), \tag{587}$$

which is the desired result. An explicit demonstration of this procedure for computing renormalized masses of scalars and fermions can be found in [207].

这正是我们想要的结果。文献 [207] 中给出了该方法计算标量和费米子重整化质量的具体示例。

Since for vacuum shift the subtleties arise in sector with $L_0^+ = 0$, we could simplify the presentation by working with an effective action where we have fully integrated out the fields with $L_0^+ \neq 0$, arriving at 1PI effective action of $L_0^+ = 0$ states only. Similarly for mass renormalization, subtleties arise in the sector with $\widehat{M}^2 = m^2$ and $k^2 + m^2 = \mathcal{O}(g_s)$. Therefore, one could work with an effective action where we integrate out all fields other than in these sectors.

由于真空位移的微妙之处出现在 $L_0^+=0$ 对应的能区，我们可以通过有效作用量简化推导: 将所有带 $L_0^+\neq 0$ 的场完全积分掉，最终得到仅含 $L_0^+=0$ 态的 1PI 有效作用量。类似地，质量重整化的微妙之处出现在带 $\widehat{M}^2=m^2$ 和 $k^2+m^2=\mathcal{O}(g_s)$ 的能区，因此我们可以只保留这两个能区的场，将其他场全部积分掉得到有效作用量。

D-Instanton Contribution to String Amplitudes

D 瞬子对弦振幅的贡献

D-instantons are D-branes with Dirichlet boundary conditions on all directions in space-time including (Euclidean) time. A Dp brane extends over p spatial dimensions and time. A D0 brane is thus a point in space, extending over all time. A D-instanton can be thought of as a D(-1) brane. It has a finite action, which can be obtained either by analyzing the partition function on an annulus with boundaries lying on the instanton [30] or by formulating an open string field theory on the background of a Dp brane with $p\geq 0$ and finding a soliton solution representing the instanton. The action $\mathcal{T}$ of that solution is the D-instanton action, and it is of the form $\mathcal{T}=A/g_s$ where A is independent of the string coupling. As a result, the D-instanton gives a non-perturbative contribution to the string amplitudes carrying an overall factor $e^{-\mathcal{T}}$. This is to be contrasted with the instanton corrections in quantum field theories, which are suppressed by e^{-B/g_s^2} for some g_s independent constant B .

D 瞬子是在时空所有方向 (包括欧几里得时间) 都满足狄利克雷边界条件的 D 膜。一个 Dp 膜覆盖 p 个空间维度和时间，因此 D0 膜是空间中一个点，延伸覆盖全部时间。D 瞬子可以看作是 D(-1) 膜，它具有有限作用量，既可以通过分析边界落在瞬子 [30] 上的环面配分函数得到，也可以通过在带有 $p\geq 0$ 的 Dp 膜背景下构造开弦场论，找到代表该瞬子的孤子解得到。该解的作用量 $\mathcal{T}$ 就是 D 瞬子作用量，形式为 $\mathcal{T}=A/g_s$ ，其中 A 与弦耦合常数无关。因此，D 瞬子对弦振幅给出非微扰贡献，整体带有因子 $e^{-\mathcal{T}}$ 。这与量子场论中的瞬子修正形成对比，后者的修正被 e^{-B/g_s^2} 压低，其中 g_s 是与耦合无关的常数 B 。

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[30] In CFT, the action of the instanton can be obtained by looking at the amplitude for an annulus with D-instanton boundary conditions and examining the factorization in the closed string channel, a procedure used to calculate the tension of branes [210,211]. The familiar formula for the tension of a Dp brane, evaluated for $p=-1$, gives the instanton "tension," which is its action.

[30] 在共形场论中，瞬子作用量可以通过研究带有 D 瞬子边界条件的环面振幅，分析其闭弦道的因子化得到，这个过程原本用于计算膜的张力 [210,211]。将我们熟悉的 Dp 膜张力公式应用于 $p=-1$ ，就能得到瞬子的“张力”，也就是它的作用量。

In our analysis, we shall use a slightly generalized notion of D-instanton where we allow Neumann boundary condition along some of the compact directions of space-time. These are often referred to as compact Euclidean D-branes. They carry finite action given by the tension of the brane integrated along the D-brane world-volume and play the same role in string theory as ordinary D-instantons. Furthermore, the analysis of the contribution from these amplitudes encounters the same subtleties as D-(-1)-branes, and hence these systems can be discussed together.

在本文的分析中，我们采用稍微推广的 D 瞬子概念，允许其在时空部分紧致方向上满足诺依曼边界条件。这类对象通常被称为紧致欧几里得 D 膜，它们具有有限作用量，等于膜张力沿 D 膜世界体积的积分，在弦论中发挥和普通 D 瞬子相同的作用。此外，分析这类振幅贡献时遇到的微妙问题和 D(-1) 膜相同，因此可以将这些系统放在一起讨论。

Any generic amplitude in string theory will receive D-instanton contribution, but they are particularly important if the amplitude under consideration vanishes in perturbation theory. Examples are amplitudes in type IIB string theory, which respect shift symmetry of the RR scalar field in perturbation theory, but this symmetry is violated by D-instanton-induced amplitudes. Another example involves type IIA or type IIB string theory on Calabi-Yau orientifolds where certain terms in the superpotential are prevented from arising in perturbation theory due to the presence of shift symmetry of some scalars but can arise due to D-instanton effects.

弦论中任何一般振幅都会受到 D 瞬子贡献，但当所研究的振幅在微扰论中为零时，D 瞬子的贡献尤其重要。例子包括 IIB 型弦论中的振幅: 这类振幅在微扰论中满足 RR 标量场的平移对称性，但 D 瞬子诱导的振幅会破坏该对称性。另一个例子是卡拉比-丘定向轨形上的 IIA 型或 IIB 型弦论: 超引力中的某些项由于部分标量的平移对称性无法在微扰论中产生，但可以通过 D 瞬子效应产生。

As for all D-branes, the excitations on the D-instanton are open strings with boundaries lying on the instanton. Since instantons are transient objects, however, these open strings are not allowed asymptotic states. Like the modes of an ordinary instanton, they describe fluctuations in (string) fields around a non-trivial saddle point and must be integrated out. The allowed asymptotic states in the theory could be closed strings and/or open strings on other D-branes whose world-volume extends along the time direction.

和所有 D 膜一样，D 瞬子上的激发是边界落在瞬子上的开弦。但由于瞬子是瞬态对象，这些开弦不存在渐近态。和普通瞬子的模式一样，它们描述非平凡鞍点附近 (弦) 场的涨落，必须被积掉。该理论中允许的渐近态可以是闭弦，或是世界体积沿时间方向延伸的其他 D 膜上的开弦。

In principle, the D-instanton contribution to an amplitude is obtained by summing over world-sheet diagrams where we allow the world-sheet to have boundaries, with D-instanton boundary condition at the boundaries. In practice, such computation runs into divergences from the boundaries of the moduli spaces of Riemann surfaces. As we shall explain below, these divergences have similar origin as the divergences that arise in the study of mass renormalization and vacuum shift.

原则上，振幅的 D 瞬子贡献可以通过对世界面图求和得到: 我们允许世界面带有边界，边界满足 D 瞬子边界条件。实际计算中，这类计算会遇到来自黎曼曲面模空间边界的发散。我们下文会说明，这些发散的起源和研究质量重整化与真空平移时出现的发散起源类似。

Since string field theory is designed to formally reproduce the world-sheet results, we can use the appropriate open-closed string field theory to compute D-instanton induced amplitudes. This string field theory has an unusual feature: since the open strings on the D-instanton have Dirichlet boundary condition along all non-compact directions, they do not carry any momentum along non-compact directions. Therefore, they represent discrete modes rather than fields. Among them are zero modes with zero L_0 eigenvalues. Therefore, in the Siegel gauge, their propagator diverges, and whenever such modes propagate as an internal state in a Feynman diagram, we get a divergent result. As usual, in the world-sheet formalism, these divergences arise whenever the Riemann surface degenerates and there is no systematic procedure to regulate them and extract an unambiguous finite answer. In string field theory, however, we can draw insights from quantum field theory and try to follow the same rules that we use while dealing with the zero modes of ordinary instantons.

由于弦场论本就是为正式重现世界面结果设计的，因此我们可以用适配的开-闭弦场论来计算 D 瞬子诱导的振幅。这套弦场论有一个特殊性质: 由于 D 瞬子上的开弦在所有非紧致方向上都满足狄利克雷边界条件，它们在非紧致方向上不携带任何动量。因此，它们代表的是离散模式而非场。其中就包含本征值为零的 L_0 零模。因此在西格尔规范中，这些零模的传播子发散，并且只要这类模式作为内部态出现在费曼图中传播，我们就会得到发散结果。和往常一样，在世界面形式体系中，这些发散只要黎曼曲面退化就会出现，且没有系统的方法可以对它们正规化，从而提取出明确有限的结果。但在弦场论中，我们可以从量子场论获取启发，遵循我们处理普通瞬子零模时沿用的相同规则来解决问题。

It turns out that there are two issues in this analysis that need separate discussion: first, the overall normalization of the amplitude, affecting even the leading order result, and, second, the higher-order corrections to the amplitude. We shall discuss them in turn. While carrying out this analysis, we need to keep in mind some specific features of D-instanton contributions to amplitudes that are not present in open-closed string field theory amplitudes in the presence of D-branes that extend along the time direction.

研究发现，该分析存在两个需要分开讨论的问题: 其一，振幅的整体归一化会影响到领头阶结果；其二，振幅的高阶修正。我们将依次对这两个问题展开讨论。在开展该分析的过程中，我们需要牢记 D 瞬子对振幅贡献的若干特殊性质，这些性质并不存在于沿时间方向延伸的 D 膜存在情况下的开-闭弦场论振幅中。

1. All D-instanton amplitudes are multiplied by a factor of $e^{-\mathcal{T}}$, where $\mathcal{T}$ is the ”tension” of the D-instanton. This is the usual suppression factor that accompanies the contribution from the non-perturbative saddle points in a path integral. From the world-sheet perspective, this can be viewed as the exponential of the disk partition function (with Dirichlet boundary condition), but this will not be important for our analysis.

1. 所有 D 瞬子振幅都乘有因子 $e^{-\mathcal{T}}$ ，其中 $\mathcal{T}$ 是 D 瞬子的“张力”。这是路径积分中非微扰鞍点贡献通常伴随的压低因子。从世界面视角来看，这可以看作圆盘配分函数 (带狄利克雷边界条件) 的指数，但这对我们的分析并不重要。

2. The space-time translation invariance is broken by the Dirichlet boundary condition on the open string along the non-compact directions. Therefore, individual world-sheet diagrams do not conserve space-time momenta, although, as will be discussed later, at the end, we recover momentum conservation after proper treatment of the zero modes. Non-conservation of the momenta by individual world-sheet diagrams means that disconnected world-sheets are on the same footing as the connected ones. This is to be contrasted with the world-sheet diagrams associated with closed string theory or open-closed string theory in the presence of space filling D-branes, where disconnected world-sheets have separate momentum conservation on each connected component, and as a result such world-sheets contribute only in a codimension one or higher subspace of the full kinematic space.

2. 开弦沿非紧致方向满足狄利克雷边界条件，破坏了时空平移不变性。因此，单独的世界面图不满足时空动量守恒，不过正如后续讨论所示，最终我们对零模做适当处理后可以恢复动量守恒。单个世界面图动量不守恒意味着不连通世界面与连通世界面地位平等。这一点与填充空间 D 膜存在下的闭弦理论或开-闭弦理论的世界面图形成对比: 在后者中，不连通世界面的每个连通分支分别满足动量守恒，因此这类世界面仅在整个运动学空间的余维数为 1 或更高的子空间中才有贡献。

Overall Normalization

总归一化

Let us consider the D-instanton contribution to an amplitude for a fixed set of external closed strings. Since each disk is accompanied by a factor of $1/g_s$ and each annulus is accompanied by a factor of $g_s^0 = 1$, the leading contribution to an amplitude with n external closed strings comes from n disconnected disks, each carrying a single closed string insertion and arbitrary number of annuli with no insertions. Note that disks with no insertion are already counted in the $e^{-\mathcal{T}}$ factor.

我们来考虑 D 瞬子对一组固定外闭弦振幅的贡献。由于每个圆盘带有一个因子 $1/g_s$ ，每个环带带有一个因子 $g_s^0 = 1$ ，因此含 n 个外闭弦的振幅的领头贡献来自 n 个不连通圆盘，每个圆盘插入单个闭弦，还可以包含任意数量无插入的环带。注意无插入圆盘已经算入 $e^{-\mathcal{T}}$ 因子中了。

The sum over different number of annuli can be exponentiated to give the exponential of the annulus partition function. This is a universal factor that appears as an overall multiplicative factor in all amplitudes in a given instanton sector. However, the world-sheet expression for the annulus partition function often diverges. Our goal in this section will be to identify the origin of these divergences and discuss how to rectify them with the help of string field theory.

对不同数量环带的求和可以指数化，得到环带配分函数的指数。这是一个普适因子，作为整体乘性因子出现在给定瞬子扇区的所有振幅中。然而，环带配分函数的世界面表达式通常存在发散。本节我们的目标是找出这些发散的起源，并讨论如何借助弦场论修正它们。

For simplicity, we shall focus on bosonic string theory, but the formalism can be generalized to superstring theories as well [212-216]. Generically, the exponential of the annulus partition function, denoted by N , takes the form

为简单起见，我们将聚焦于玻色弦理论，但该形式论也可以推广到超弦理论 [212-216]。一般来说，环带配分函数的指数记为 N ，形式如下

$$N \equiv \exp\left[\int_0^\infty \frac{dt}{t} F(t)\right], \tag{588}$$

where $F(t)$ has the form

其中 $F(t)$ 形式为

$$F(t) = \frac{1}{2}\,\mathrm{Tr}\left\{(-1)^f e^{-2\pi t L_0}\right\}, \tag{589}$$

with the trace running over all the Siegel gauge open string states on the D-instanton. $(-1)^f$ denotes the Grassmann parity of the open string field component multiplying the state in the expansion of the string field. The factor of two in the denominator in (589) appears because the annulus has a discrete Z_2 symmetry under which world-sheet coordinates change sign. The Siegel gauge condition on the states is needed because the annulus has zero modes of b and c ghosts that must be removed, reflecting the fixing of the translation symmetry on the world-sheet.

迹遍历 D 瞬子上所有西格尔规范的开弦态。$(-1)^f$ 表示开弦场分量在弦场展开中乘态时的格拉斯曼奇偶性。式 (589) 分母中的因子 2 出现的原因是，环带具有一个分立 Z_2 对称性，世界面坐标在该变换下变号。对态施加西格尔规范条件是必要的，因为环带存在 b 和 c 鬼的零模，这些零模必须被移除，这反映了世界面上平移对称性的规范固定。

If $|\phi_b\rangle$ and $|\phi_f\rangle$ denote a basis of states in the Siegel gauge, multiplying Grassmann even and Grassmann odd string fields, respectively, and h_b and h_f represent the conformal weights of $|\phi_b\rangle$ and $|\phi_f\rangle$, respectively, then we can express (588) as

如果 $|\phi_b\rangle$ 和 $|\phi_f\rangle$ 分别是西格尔规范中格拉斯曼偶和格拉斯曼奇弦场的态基，h_b 和 h_f 分别表示 $|\phi_b\rangle$ 和 $|\phi_f\rangle$ 的共形权重，那么我们可以将 (588) 表示为

$$N = \exp\left[\int_0^\infty \frac{dt}{2t}\left(\sum_b e^{-2\pi t h_b} - \sum_f e^{-2\pi t h_f}\right)\right]. \tag{590}$$

The integral over t can have divergences from the $t = 0$ and the $t = \infty$ ends of the integral. The divergence from the $t = 0$ end can be attributed to the closed string channel, either from closed string tachyons or from massless closed string states. Such divergences, if present, represent genuine problems with the theory or with the choice of vacuum of the theory. In the following, we shall assume that such divergences are absent. Examples of such cases are critical bosonic and superstring theories in more than two non-compact dimensions. The divergence from the $t \to \infty$ end is associated with light open string states on the D-instanton and will be the focus of our analysis.

对 t 的积分可能在积分的 $t = 0$ 端和 $t = \infty$ 端产生发散。$t = 0$ 端的发散可以归因于闭弦道，来源要么是闭弦快子，要么是无质量闭弦态。这类发散如果存在，代表理论本身或理论真空选择存在真实问题。在下文中我们假设这类发散不存在，临界玻色弦理论和超弦理论在超过两个非紧致维度的情况就是这类例子。$t \to \infty$ 端的发散与 D 瞬子上的轻开弦态相关，将是我们分析的核心。

The absence of a divergence in the $t \to 0$ limit implies that we have equal number of Grassmann even and Grassmann odd modes, in an appropriately regulated sense. We can now use the identity

$t \to 0$ 极限下没有发散意味着，在适当正规化的意义下，格拉斯曼偶模和格拉斯曼奇模的数量相等。我们现在可以利用恒等式

$$\int_0^\infty \frac{dt}{2t}\left(e^{-2\pi h_b t} - e^{-2\pi h_f t}\right) = \ln\sqrt{\frac{h_f}{h_b}}, \tag{591}$$

to conclude that

得到结论

$$N = \prod_{b,f} \sqrt{\frac{h_f}{h_b}} = (\mathrm{sdet}\,(L_0))^{-1/2}. \tag{592}$$

This is turn can be given the following path integral representation in the open string field theory on the D-instanton. Let $\{|\phi_r\rangle\}$ be a basis of open string states in the Siegel gauge, normalized so that

反过来，可以在 D 瞬子的开弦场论中给出如下路径积分表示。令 $\{|\phi_r\rangle\}$ 为西格尔规范中开弦态的一组基，归一化满足

$$\mathrm{sdet}\, M = 1\,, \text{with } \; M_{rs} \equiv \langle \phi_r\, |c_0|\, \phi_s\rangle'\,. \tag{593}$$

Since open strings living on the D-instanton do not carry any continuous momentum, the indices r, s are discrete. We can now give a "path integral" description of N using the identities

由于 D 瞬子上的开弦不携带连续动量，指标 r, s 是离散的。我们现在可以利用恒等式对 N 给出“路径积分”描述

$$\int \frac{d\phi_b}{\sqrt{2\pi}} e^{-\frac{1}{2}h_b\phi_b^2} = h_b^{-1/2}, \int dp_f dq_f e^{-h_f p_f q_f} = h_f, \tag{594}$$

holding for Grassmann even ϕ_b and Grassmann odd p_f and q_f . If we expand the Siegel gauge open string field as

该式对格拉斯曼偶的 ϕ_b 以及格拉斯曼奇的 p_f 和 q_f 成立。如果我们将西格尔规范开弦场展开为

$$|\psi_o\rangle = \sum_r |\phi_r\rangle\, \psi_r \tag{595}$$

one can confirm that

可以验证

$$N = (\mathrm{sdet}\,(L_0))^{-1/2} = \int \prod_r D\psi_r \exp\left[\frac{1}{2}\langle\psi_o\, |c_0 L_0|\, \psi_o\rangle'\right], \tag{596}$$

where the integration measure $D\psi_r$ is defined to be $d\psi_r/\sqrt{2\pi}$ for Grassmann even ψ_r and $d\psi_r$ for Grassmann odd ψ_r . In writing this, we have used the result that for any fixed L_0 eigenvalue, the Grassmann odd ψ_r ’ s always come in pairs since they have even ghost number, and in the action, a ghost number n state is paired with a ghost number $(2-n)$ state.

其中积分测度 $D\psi_r$ 对格拉斯曼偶 ψ_r 定义为 $d\psi_r/\sqrt{2\pi}$ ，对格拉斯曼奇 ψ_r 定义为 $d\psi_r$ 。书写该式时，我们利用了下述结论: 对任意固定的 L_0 本征值，由于格拉斯曼奇 ψ_r 具有偶数鬼数，它们总是成对出现，且在作用量中，一个鬼数为 n 的态与一个鬼数为 $(2-n)$ 的态配对。

Note that (591) holds only when $h_b > 0$ and $h_f \neq 0$, but we shall use (596) as the definition of N even when L_0 has negative or vanishing eigenvalues, since from the perspective of string field theory, this is the expression that arises naturally from the one-loop determinant of fluctuations around the instanton in the Siegel gauge. For negative h_b , we need to take the integration contour of ϕ_b along the imaginary axis, which is in fact the steepest descent contour of ϕ_b . This gives us back $h_b^{-1/2}$. For vanishing h_b or h_f , we have open string zero modes. Integrations over these zero modes naively diverge for bosonic zero modes and vanish for fermionic zero modes, and we need to understand the physical origin of the zero modes in order to deal with them. This is what we shall discuss now.

> 注意 (591) 仅在 $h_b > 0$ 和 $h_f \neq 0$ 时成立，但我们将 (596) 用作 N 的定义，即使在 L_0 本征值为负或为零时也如此，因为从弦场论的视角来看，该式自然来自西格尔规范下瞬子周围涨落的一圈行列式。对负的 h_b ，我们需要让 ϕ_b 的积分围道沿虚轴延伸，这实际上就是 ϕ_b 的最速下降围道，最终结果仍为 $h_b^{-1/2}$ 。当 h_b 或 h_f 为零时，会存在开弦零模。对这些零模的积分 naive 来看对玻色零模发散，对费米零模为零，我们需要理解零模的物理起源才能处理它们，这正是我们接下来要讨论的内容。

In the Siegel gauge, there are two types of zero modes. Let us begin with the first type. If we denote by α_n^μ the oscillators associated with the flat directions transverse to the D-instanton, normalized so that $\left[\alpha_m^\mu, \alpha_n^\upsilon\right] = m\eta^{\mu\upsilon}\delta_{m+n,0}$, then we have $L_0 = 0$ states of the form $c_1\alpha_{-1}^\mu|0\rangle$. If we denote by ξ_μ the coefficients multiplying these states in the expansion of the open string field, then ξ_μ ’ s represent Grassmann even zero modes. Physically, these are related to the collective coordinates of the

> 在西格尔规范中存在两类零模。我们先讨论第一类。若用 α_n^μ 表示与垂直于 D 瞬子的平坦方向关联的振子，归一化满足 $\left[\alpha_m^\mu, \alpha_n^\upsilon\right] = m\eta^{\mu\upsilon}\delta_{m+n,0}$ ，则我们得到形式为 $c_1\alpha_{-1}^\mu|0\rangle$ 的 $L_0 = 0$ 态。若用 ξ_μ 表示开弦场展开中乘在这些态前的系数，则 ξ_μ 代表格拉斯曼偶零模。物理上，它们对应下述集体坐标:

D-instanton that correspond to shift δy^μ of the D-instanton position y^μ along the transverse directions. The precise relation between ξ^μ and δy^μ may be found by comparing the coupling of ξ^μ to a set of closed strings carrying total momentum p to the expected coupling of these closed strings to y^μ via the $e^{ip\cdot y}$ term. At the leading order, the coupling of ξ^μ to closed strings is via the disk amplitude, and the comparison yields [212, 217]

> 对应 D 瞬子位置 y^μ 沿横向平移 δy^μ 的 D 瞬子集体坐标。ξ^μ 与 δy^μ 的精确关系可以通过下述方式得到: 将 ξ^μ 与总动量为 p 的一组闭弦的耦合，和这些闭弦通过 $e^{ip\cdot y}$ 项与 y^μ 的预期耦合对比。领头阶下，ξ^μ 与闭弦的耦合通过圆盘振幅实现，对比可得文献 [212, 217]

$$\xi^\mu = \frac{1}{\sqrt{2}\pi g_o}\delta y^\mu, \tag{597}$$

where g_o is the open string coupling in the instanton open string field theory, related to its action $\mathcal{T}$ via $\mathcal{T} = 1/\left(2\pi^2 g_o^2\right)$. As in the case of collective modes of the instanton in a quantum field theory, we need to carry out integration over these modes at the end. Even though the annulus amplitude does not depend on the D-instanton position, other components of the world-sheet involving external closed strings carrying momentum insertions do depend on y through the $e^{ip\cdot y}$ terms. Therefore, after changing variables from ξ^μ to y^μ , the integration over these zero modes will produce the usual momentum conserving delta function $(2\pi)^D\delta^{(D)}\left(\sum_i p_i\right)$ where D is the number of non-compact directions.

其中 g_o 是瞬子开弦场论中的开弦耦合，它通过 $\mathcal{T} = 1/\left(2\pi^2 g_o^2\right)$ 与其作用量 $\mathcal{T}$ 相关。和量子场论中瞬子集体模的情况一样，我们最终需要对这些模进行积分。尽管环形振幅不依赖 D 瞬子位置，但其他包含携带动量插入的外闭弦的世界面分量确实会通过 $e^{ip\cdot y}$ 项依赖于 y 。因此，将变量从 ξ^μ 换为 y^μ 后，对这些零模的积分会得到我们熟悉的动量守恒 δ 函数 $(2\pi)^D\delta^{(D)}\left(\sum_i p_i\right)$ ，其中 D 是非紧致方向的数目。

The other type of zero modes arises from the $L_0 = 0$ Siegel gauge states $|0\rangle$ and $c_1c_{-1}|0\rangle$. Since these states carry even ghost number, the coefficients p, q multiplying these states in the expansion of the string field are Grassmann odd:

另一类零模来自 $L_0 = 0$ 西格尔规范态 $|0\rangle$ 和 $c_1c_{-1}|0\rangle$ 。由于这些态携带偶数鬼数，弦场展开中乘在这些态前的系数 p, q 是格拉斯曼奇的:

$$|\psi_o\rangle = p\,|0\rangle + qc_1c_{-1}|\,0\rangle + \cdots . \tag{598}$$

To find the physical origin of these modes, let us deform the theory so that the vacuum has conformal weight h . This can be achieved, e.g., by putting slightly different boundary conditions on the two boundaries attached to the instanton or considering a Dp -brane instead of D -instanton so that we can consider momentum carrying states. [31] For definiteness, we shall assume that the two boundaries attached to the instanton are displaced in the 0-direction by a small amount to produce the weight h of the vacuum. With the dimension h vacuum and the string field (598), the Euclidean action has a term

为了找到这些模的物理起源，我们对理论做形变，使真空具有共形权重 h 。这可以通过多种方式实现，例如对连接瞬子的两个边界施加略有不同的边界条件，或是考虑 Dp 膜而非 D 瞬子，从而可以研究携带动量的态。[31] 为明确起见，我们假设连接瞬子的两个边界在 0 方向上有微小位移，使得真空获得权重 h 。在具有维度 h 的真空和弦场 (598) 下，欧几里得作用量包含一项

$$\frac{1}{2}\,\langle\psi_o\,|c_0L_0|\,\psi_o\rangle' = -hpq + \cdots , \tag{599}$$

and integration over p, q will produce a factor of h as part of $(\mathrm{sdet}\,(L_0))^{1/2}$.

对 p, q 的积分会给出因子 h ，作为 $(\mathrm{sdet}\,(L_0))^{1/2}$ 的一部分。

To find the physical interpretation of the modes p, q , let us consider the classical open string field theory before gauge fixing. In this theory, fields p, q are absent since they multiply string states of ghost number other than one, but there are additional string field components that multiply states of ghost number one that do not satisfy the Siegel gauge condition. Let us suppose that ϕ is the (bosonic) mode that multiplies the out of Siegel gauge state $c_0|0\rangle$ so that the string field contains the term

为了找到模 p, q 的物理解释，我们考虑规范固定前的经典开弦场论。该理论中不存在场 p, q ，因为它们乘的是鬼数不为 1 的弦态，但存在额外的弦场分量，它们乘鬼数为 1、不满足西格尔规范条件的态。假设 ϕ 是乘在西格尔规范外态 $c_0|0\rangle$ 上的 (玻色) 模，因此弦场包含项

$$|\psi_o\rangle = i\phi c_0|0\rangle + \cdots \tag{600}$$

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[31] The purpose of this deformation is to see how string field theory and the world-sheet approach give the same results in the absence of zero modes. Eventually, we shall set $h = 0$ where the world-sheet formalism breaks down, but string field theory still gives well-defined results.

> [31] 该形变的目的是验证在不存在零模时，弦场论和世界面方法如何给出一致的结果。最终我们会取 $h = 0$，此时世界面形式失效，但弦场论仍能给出定义良好的结果。

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before gauge fixing. With a bosonic parameter θ, we have $Q\theta\,|0\rangle = h\theta c_0|\,0\rangle$, so that under a gauge transformation with gauge parameter $i\theta|0\rangle$, we get

> 在规范固定前。给定玻色参数 θ，我们有 $Q\theta\,|0\rangle = h\theta c_0|\,0\rangle$，因此在带规范参数 $i\theta|0\rangle$ 的规范变换下，我们得到

$$\delta\phi = h\theta. \tag{601}$$

Therefore, if we fix the gauge by setting $\phi = 0$, as in Siegel gauge, we shall get a Jacobian h, which we are supposed to represent as integral over Fadeev-Popov ghosts. Since integration over p, q produces precisely this factor, we see that p, q has the interpretation of Fadeev-Popov ghosts that arise from fixing the gauge transformation generated by $i\theta|0\rangle$ by setting $\phi = 0$. Therefore, in the gauge invariant form of the path integral, the integration over p, q can be replaced by

> 因此，如果我们像西格尔规范那样通过设定 $\phi = 0$ 固定规范，我们会得到雅可比因子 h，该因子需要表示为对法捷耶夫-波波夫鬼的积分。由于对 p, q 的积分恰好给出这个因子，可见 p, q 就是法捷耶夫-波波夫鬼，它来自通过设定 $\phi = 0$ 固定由 $i\theta|0\rangle$ 生成的规范变换。因此，在路径积分的规范不变形式中，对 p, q 的积分可以替换为

$$\int dpdq \to \int d\phi e^{S_{\phi,\xi^0}} / \int d\theta. \tag{602}$$

Here S_{ϕ,ξ^0} now stands for the part of the gauge-invariant Euclidean action that depends on ϕ and ξ^0 :

> 此处 S_{ϕ,ξ^0} 表示规范不变欧几里得作用量中依赖于 ϕ 和 ξ^0 的部分:

$$S_{\phi,\xi^0} = \frac{1}{2}\langle\psi_o\,|Q|\,\psi_o\rangle\Big|_{\phi,\xi^0} = -\left(\phi - \sqrt{\frac{h}{2}}\xi^0\right)^2. \tag{603}$$

This is invariant under the gauge transformation

> 这在规范变换下不变。

$$\delta\phi = h,\ \delta\xi^0 = \sqrt{2h}. \tag{604}$$

Note that ξ^0 plays a special role since we have displaced the boundary conditions at the two ends of the open string along the zeroth direction. The other ξ^i ' s will just contribute gauge-invariant $h(\xi^i)^2/2$ terms. Eventually, when we take $h \to 0$ limit, this special role of the 0 direction will disappear.

请注意，ξ^0 发挥特殊作用，因为我们将开弦两端的边界条件沿第零方向平移了。其他 ξ^i 仅贡献规范不变的 $h(\xi^i)^2/2$ 项。最终，当我们取 $h \to 0$ 极限时，第零方向的这种特殊作用就会消失。

For $h = 0$, the gauge-fixing procedure breaks down: since ϕ no longer transforms under the gauge transformation generated by $\theta, \phi = 0$ is not a good choice of gauge. This is reflected in the vanishing of the Jacobian factor h . However, the gauge invariant form of the path integral given on the right-hand side of (602) still makes sense, and we shall use this to compute N . The integration over ϕ now produces

对于 $h = 0$，规范固定过程失效: 由于 ϕ 不再在 $\theta, \phi = 0$ 生成的规范变换下变换，这不是一个合适的规范选择。这体现在雅可比因子 h 变为零。不过，(602) 右侧给出的路径积分的规范不变形式仍然有意义，我们将用它来计算 N 。现在对 ϕ 积分得到

$$\int d\phi e^{-\phi^2} = \sqrt{\pi}. \tag{605}$$

To calculate $\int d\theta$, we note that the gauge transformation parameter θ is the analog of the usual $U(1)$ gauge transformation parameter on a D-brane; however, since open strings on a D-instanton do not carry momentum, the gauge transformation is rigid. Let α be the parameter of this rigid $U(1)$ transformation, normalized such that an open string ending on the D-instanton picks up a phase $e^{i\alpha}$ under this transformation. Therefore, α has period 2π . Comparing the infinitesimal version of this gauge transformation law to the string field theory gauge transformation by the parameter $i\theta|0\rangle$, we can find the relation between θ and α . At the leading order, this takes the form [217]

为了计算 $\int d\theta$，我们注意到规范变换参数 θ 是 D 膜上通常 $U(1)$ 规范变换参数的类比；然而，由于 D 瞬子上的开弦不携带动量，该规范变换是刚性的。设 α 为这个刚性 $U(1)$ 变换的参数，归一化条件为: 终止于 D 瞬子的开弦在该变换下获得相位 $e^{i\alpha}$ 。因此，α 具有周期 2π 。将该规范变换律的无穷小版本与参数为 $i\theta|0\rangle$ 的弦场论规范变换对比，我们可以找到 θ 和 α 之间的关系。在领头阶，其形式为 [217]

$$\theta = \alpha / g_o. \tag{606}$$

This gives

由此可得

$$\int d\theta = 2\pi / g_o. \tag{607}$$

Equations (597), (602), (605), and (607) now show that the integration over the zero modes ξ^μ, p, q can be replaced by

式 (597)、(602)、(605) 和 (607) 表明，对零模 ξ^μ, p, q 的积分可以替换为

$$\int dpdq \prod_\mu \frac{d\xi^\mu}{\sqrt{2\pi}} \to \left(2\pi\sqrt{\pi}g_o\right)^{-d} \frac{g_o}{2\pi}\sqrt{\pi} \int \prod_\mu dy^\mu. \tag{608}$$

As already mentioned, the integration over y^μ is carried out at the end and produces the momentum conserving delta function. The integration over the $L_0 > 0$ modes of the string field is Gaussian integrals that can be performed explicitly using (594). Therefore, the normalization constant may be written as

如前所述，对 y^μ 的积分在最后完成，得到动量守恒 δ 函数。对弦场 $L_0 > 0$ 模的积分是高斯积分，可以利用 (594) 显式计算。因此，归一化常数可以写为

$$N = \left(2\pi\sqrt{\pi}g_o\right)^{-d} \frac{g_o}{2\pi}\sqrt{\pi} \prod_b{}' h_b^{-1/2} \prod_f{}' h_f^{1/2} (2\pi)^d \delta^{(d)}\left(\sum_i p_i\right), \tag{609}$$

where the primes on the products indicate that zero modes are excluded from this product.

其中乘积上的撇表示该乘积中排除零模。

This formula involves infinite products of h_b ’ s and h_f ’ s and is not always easy to use unless there are cancellations. We can simplify this by using (591) in reverse, expressing the infinite product as exponentials of an integral for modes with $h_b, h_f > 0$. However, we need equal number of Grassmann even and Grassmann odd modes to apply (591) since (591) pairs a bosonic mode with a fermionic mode. This can be achieved by dividing the set ’ into two sets: ” containing a finite number of h_b ’ s and h_f ’ s that include all tachyonic modes and possibly some $L_0 > 0$ modes and ”” containing infinite but equal numbers of strictly positive h_b ’ s and h_f ’ s. We can then write

该公式包含 h_b 和 h_f 的无穷乘积，除非存在抵消，否则通常并不易用。我们可以反过来利用 (591) 对其化简，将无穷乘积表示为 $h_b, h_f > 0$ 模积分的指数形式。但应用 (591) 要求格拉斯曼偶模和格拉斯曼奇模数量相等，因为 (591) 将一个玻色模和一个费米模配对。我们可以通过将集合’ 拆分为两个集合来实现这一点:” 包含有限个 h_b 和 h_f ，覆盖所有快子模，还可能包含部分 $L_0 > 0$ 模，”” 则包含无穷多个严格正的 h_b 和 h_f ，且二者数量相等。于是我们可以写出

$$\prod_b h_b^{-1/2} \prod_f h_f^{1/2}$$
$$= \prod_b{}'' h_b^{-1/2} \prod_f{}'' h_f^{1/2} \exp\left[-\int_0^\infty \frac{dt}{2t}\left(\sum_b{}''' e^{-2\pi t h_b} - \sum_f{}''' e^{-2\pi t h_f}\right)\right].$$

(610)

The result can be shown to be independent of how we divide the set ’ into ” and ””.

可以证明结果与我们如何拆分集合’ 为” 和”” 无关。

A final comment on the normalization factor: sometimes, the actual integration contour over the open string modes on the D-instanton does not include the full steepest descent contour spanning the range $(-\infty, \infty)$

in (594) but only a faction f of the steepest descent contour [217]. This usually happens in the presence of tachyonic modes. In such cases, the normalization constant N contains an extra factor of f .

关于归一化因子的最后一点说明: 有时，D 瞬子上开弦模的实际积分围道不覆盖 (594) 中跨越范围 $(-\infty, \infty)$ 的完整最速下降围道，仅取最速下降围道的一部分 f [217]。这种情况通常出现在存在快子模时。在此类情况下，归一化常数 N 会多出一个因子 f 。

The analysis of the normalization for D-instanton amplitudes in superstring theory is similar. The only new feature is that besides having the bosonic zero modes associated with broken translation symmetry, we now also have fermionic zero modes associated with broken supersymmetry. We treat them in the same way as the bosonic zero modes, separating out their contribution and integrating over them at the end. The last step requires inserting vertex operators of the fermionic zero modes, arising in the open string sector, into the correlation functions as if they are external states, even though physically these amplitudes represent scattering amplitudes of external closed string states only [212,218,219].

超弦理论中 D 瞬子振幅的归一化分析是类似的。唯一的新特点是，除了存在与破缺平移对称性关联的玻色零模外，我们现在还拥有与破缺超对称关联的费米零模。我们对费米零模采用与玻色零模相同的处理方式: 分离出它们的贡献，最后再对其积分。最后一步需要将开弦 sector 产生的费米零模顶点算子插入关联函数，就好像它们是外态一样，尽管物理上这些振幅仅代表外闭弦态的散射振幅 [212,218,219]。

The procedure outlined in this section has been tested against known results from dual descriptions in various examples [212,214,215,217,220-223].

本节概述的方法已经在多个例子中通过对偶描述的已知结果得到了验证 [212,214,215,217,220-223]。

Higher-Order Corrections to the Instanton Amplitudes

瞬子振幅的高阶修正

When we consider higher-order corrections to the instanton amplitudes, we get additional divergences from degenerate Riemann surfaces. In the language of string field theory, these divergences come from on-shell/tachyonic internal propagators in Feynman diagrams and can be dealt with by using standard quantum field theory methods. For example, when there is open string degeneration, divergences are associated with open string zero modes or tachyonic modes propagating in the intermediate channel. For the zero modes, we simply remove their contribution to the propagator by hand since we carry out path integral over zero modes at the end, as discussed in section ”Overall Normalization”. For the tachyonic modes, we replace the world-sheet contribution to their propagator $\int dte^{-2\pi t L_0}$ by $1/(2\pi L_0)$ even for negative L_0 . This produces manifestly finite results. However, to implement this in practice, we need to first express the amplitude as a sum over Feynman diagrams in string field theory, so that we can identify which part of the amplitude is coming from the zero mode or tachyonic mode exchange diagrams and remove only that part from the amplitude. As usual, this depends on the choice of the local coordinates at the punctures used to define the vertex, but the final result is expected to be independent of this choice.

当我们研究瞬子振幅的高阶修正时，简并黎曼曲面会带来额外发散。在弦场论的语言中，这些发散来自费曼图中的在壳/快子内传播子，可以用标准量子场论方法处理。例如，当存在开弦简并时，发散与中间通道传播的开弦零模或快子模相关。对于零模，正如“整体归一化”章节讨论的，由于我们最终会对零模做路径积分，我们直接手动扣除它们对传播子的贡献。对于快子模，即使负的 L_0，我们也将其传播子的世界面贡献 $\int dte^{-2\pi tL_0}$ 替换为 $1/(2\pi L_0)$，这样就能得到明显有限的结果。但要实际实现这一步，我们首先需要将振幅表示为弦场论中费曼图的和，这样才能识别出振幅的哪一部分来自零模或快子模交换图，仅将该部分从振幅中扣除。和往常一样，这依赖于定义顶点时刺点处局部坐标的选择，但预期最终结果与该选择无关。

Additional contributions to the higher-order terms come from the fact that the relations (597) and (606) get modified by higher order corrections, since the coupling of the modes ξ^μ to closed strings as well as the string field theory gauge transformation laws generated by the state $\theta\,|0\rangle$ can receive higher-order corrections. | The final result for higher-order corrections is the combined effect of all these corrections.

高阶项的额外贡献来自: 关系 (597) 和 (606) 会被高阶修正修改，因为 ξ^μ 模与闭弦的耦合，以及由态 $\theta\,|0\rangle$ can receive higher-order corrections. | 生成的弦场论规范变换定律。高阶修正的最终结果是所有这些修正的共同效应。

The procedure described here has also been tested against known results from a dual description in various examples [114, 219, 224-226]. Finally, note that so far, we have only described the contribution to the amplitude due to a single D-instanton. In section ”D-Instanton-Induced Effective Action”, we shall discuss some aspects of the contribution from more than one instanton.

这里描述的步骤已经在多个例子中通过对偶描述的已知结果完成了检验 [114, 219, 224-226]。最后要注意，到目前为止，我们仅描述了单个 D-瞬子对振幅的贡献。在“D-瞬子诱导有效作用量”章节，我们将讨论多个瞬子贡献的若干方面。

D-Instanton-Induced Effective Action

D 瞬子诱导的有效作用量

D-instantons are transient objects and do not support asymptotic states. Instead, they are saddle points of the Euclidean path integral that contribute to the Wilsonian effective action of closed string field theory (open-closed string field theory if the background contains D-branes extending in the time direction). We have seen earlier that the perturbative Wilsonian effective action satisfies the quantum BV master equation, just as the parent theory does. Therefore, it is natural to ask if the Wilsonian effective action that includes D-instanton contributions also satisfies the same property [227].

D 瞬子是瞬态客体，不存在渐近态。它们实际上是欧几里得路径积分的鞍点，对闭弦场论的威尔逊有效作用量有贡献——若背景含沿时间方向延展的 D 膜，则对应开-闭弦场论。我们此前已经看到，微扰威尔逊有效作用量和原理论一样满足量子 BV 主方程。因此我们自然会问，包含 D 瞬子贡献的威尔逊有效作用量是否也满足同样的性质 [227]。

In order to address this issue, we shall integrate out the open string modes on the D-instanton, regulating the infrared divergences using open string field theory as described in sections ”Overall Normalization” and ”Higher-Order Corrections to the Instanton Amplitudes”. We consider first a single D-instanton contribution. As mentioned at the beginning of section ”D-Instanton Contribution to String Amplitudes”, there are two additional effects that we need to take into account. First, the effective action is accompanied by a constant multiplicative factor $\mathcal{N} = e^{-\mathcal{T}} N$ that includes the $e^{-\mathcal{T}}$ factor associated with the D-instanton action and the normalization factor N associated with the annulus amplitude discussed in section ”Overall Normalization”. Second, disconnected world-sheets also contribute to the D-instanton-induced effective action with the same overall normalization factor $\mathcal{N}$. Let us denote by S_1 the single D-instanton contribution to the Wilsonian effective action but associated only with a single connected world-sheet, so that it does not include the factor of $\mathcal{N}$. After taking into account the contribution from disconnected world-sheets, the net contribution to the effective action from the one-instanton sector exponentiates and can be written as

> 为了解决这个问题，我们将积分掉 D 瞬子上的开弦模式，使用开弦场论按照“整体归一化”和“瞬子振幅的高阶修正”小节所述方法对红外发散进行正规化。我们首先考虑单个 D 瞬子的贡献。如“D 瞬子对弦振幅的贡献”小节开头所述，我们还需要考虑两个额外效应。第一，有效作用量带有一个常数乘因子 $\mathcal{N} = e^{-\mathcal{T}} N$，其中包含与 D 瞬子作用量关联的 $e^{-\mathcal{T}}$ 因子，以及“整体归一化”小节讨论的与环面振幅关联的归一化因子 N。第二，不连通世界面也会对 D 瞬子诱导的有效作用量有贡献，且带有相同的整体归一化因子 $\mathcal{N}$。我们用 S_1 表示单个 D 瞬子对威尔逊有效作用量、且仅对应单个连通世界面的贡献，因此它不包含因子 $\mathcal{N}$。计入不连通世界面的贡献后，单瞬子区对有效作用量的净贡献会指数化，可写为

$$\mathcal{N}\left(e^{S_1} - 1\right) . \tag{611}$$

Let S_0 denote the perturbative contribution to the Wilsonian effective action. Then, the net contribution to the Wilsonian effective action may be written as

> 令 S_0 表示微扰对威尔逊有效作用量的贡献。那么，威尔逊有效作用量的总贡献可写为

$$S = S_0 + \mathcal{N}\left(e^{S_1} - 1\right) + \mathcal{O}\left(\mathcal{N}^2\right), \tag{612}$$

where terms of order $\mathcal{N}^2$ or higher represent the contribution from two or more instantons. Note that we have assumed for simplicity that there is only one kind of instanton, but this analysis can be easily be generalized to the cases where the theory contains different kinds of instantons.

> 其中 $\mathcal{N}^2$ 阶及更高阶项代表两个及更多瞬子的贡献。注意为简化起见我们这里假设仅存在一种瞬子，但该分析可以很容易推广到理论包含多种瞬子的情况。

We shall now verify that S given in (612) satisfies the BV master equation. First we recall that the perturbative contribution S_0 satisfies the quantum BV master equation:

> 我们现在来验证 (612) 式给出的 S 满足 BV 主方程。首先我们回顾，微扰贡献 S_0 满足量子 BV 主方程:

$$\frac{1}{2}\{S_0, S_0\} + \Delta S_0 = 0. \tag{613}$$

The action $S_0 + S_1$ will also satisfy the BV master equation since it is obtained by starting with the master action of the Wilsonian effective action of the conventional open-closed string field theory, treating the D-instantons as conventional D-branes so that there is no multiplicative factor $\mathcal{N}$ and no disconnected diagram contribution. This gives

作用量 $S_0 + S_1$ 同样满足 BV 主方程，因为它是从常规开-闭弦场论的威尔逊有效作用量的主作用量出发得到的: 我们将 D 瞬子视作常规 D 膜，因此不存在乘因子 $\mathcal{N}$，也没有不连通图贡献。由此可得

$$\frac{1}{2}\{S_0 + S_1, S_0 + S_1\} + \Delta(S_0 + S_1) = 0. \tag{614}$$

Combining (613) and (614), we get

结合 (613) 和 (614)，我们得到

$$\{S_0, S_1\} + \frac{1}{2}\{S_1, S_1\} + \Delta S_1 = 0. \tag{615}$$

Using (612), we now write

利用 (612)，我们现在写出

$$\frac{1}{2}\{S, S\} = \frac{1}{2}\{S_0, S_0\} + \mathcal{N}\{S_0, S_1\} e^{S_1} + \mathcal{O}\left(\mathcal{N}^2\right), \tag{616}$$

and

和

$$\Delta S = \Delta S_0 + \mathcal{N}\Delta e^{S_1} = \Delta S_0 + \mathcal{N}\left(\frac{1}{2}\{S_1, S_1\} + \Delta S_1\right) e^{S_1} \tag{617}$$

Using (613) and (615), we now get

利用 (613) 和 (615)，我们现在得到

$$\frac{1}{2}\{S, S\} + \Delta S = \mathcal{O}\left(\mathcal{N}^2\right). \tag{618}$$

Thus, S satisfies the BV master equation to order $\mathcal{N}$. We shall now show that the order $\mathcal{N}^2$ terms cancel once we take into account two instanton contribution to the effective action [227].

因此，S 在 $\mathcal{N}$ 阶满足 BV 主方程。我们接下来将说明，计入双瞬子对有效作用量的贡献后，$\mathcal{N}^2$ 阶项会抵消 [227]。

For two identical instantons, there can be different kinds of connected world-sheets:

对于两个全同瞬子，可能存在不同类型的连通世界面:

1. World-sheets with no boundaries on the instantons: These contribute to S_0.

1. 世界面不存在落在瞬子上的边界: 这些对 S_0 有贡献。

2. Connected world-sheets with boundaries but with all the boundaries lying on one of the two instantons: This contribution can be identified as $2S_1$, with the factor of 2 representing that the boundary can lie on one of the two D-instantons.

2. 带边界的连通世界面，且所有边界都落在两个瞬子中的一个上: 该贡献可记为 $2S_1$ ，因子 2 代表边界可以落在两个 D 瞬子中的任意一个上。

3. Connected world-sheets with boundaries with at least one boundary lying on the first instanton and at least one boundary lying on the second instanton. We denote its contribution to the effective action by S_2 .

3. 带边界的连通世界面，其中至少一个边界落在第一个瞬子上，且至少一个边界落在第二个瞬子上。我们将它对有效作用量的贡献记为 S_2 。

For the same reasons explained above for the case of S_0+S_1 , the action $S_0+2S_1+S_2$ will satisfy the standard BV master equation:

和前文针对 S_0+S_1 情况给出的原因相同，作用量 $S_0+2S_1+S_2$ 满足标准 BV 主方程:

$$\frac{1}{2}\{S_0+2S_1+S_2,S_0+2S_1+S_2\}+\Delta\left(S_0+2S_1+S_2\right)=0. \tag{619}$$

Combining this with (613) and (615), we get

结合该式与 (613) 和 (615)，我们得到

$$\{S_0,S_2\}+2\{S_1,S_2\}+\{S_1,S_1\}+\frac{1}{2}\{S_2,S_2\}=0. \tag{620}$$

The actual contribution to the Wilsonian effective action at the two instanton order differs from $S_0+2S_1+S_2$ due to the following reasons. First of all, we have to include contributions from disconnected world-sheets but at least one connected world-sheet with one boundary on the first instanton and one boundary on the second instanton. Otherwise, the contribution to the amplitude can be regarded as second-order term induced by the single instanton effective action. Second, the contribution will have to be multiplied by $\mathcal{N}^2$, where $\mathcal{N}$ is the same constant that multiplies the single instanton effective action. The net contribution to the effective action up to two instanton order is then given by

双瞬子阶对威尔逊有效作用量的实际贡献和 $S_0+2S_1+S_2$ 不同，原因如下: 首先，我们必须包含不连通世界面的贡献，但至少要有一个连通世界面，其一条边界在第一个瞬子上，另一条边界在第二个瞬子上；否则，该对振幅的贡献会被视作单瞬子有效作用量诱导的二阶项。其次，贡献需要乘以 $\mathcal{N}^2$ ，其中 $\mathcal{N}$ 是和单瞬子有效作用量相乘的同一个常数。因此，到双瞬子阶为止，对有效作用量的总贡献由下式给出:

$$S=S_0+\mathcal{N}\left(e^{S_1}-1\right)+\frac{1}{2}\mathcal{N}^2\left(e^{S_2}-1\right)e^{2S_1}+\mathcal{O}\left(\mathcal{N}^3\right), \tag{621}$$

The factor of 1/2 is a reflection of the fact that the exchange of two identical instantons do not generate a new contribution. This symmetry is not captured in S_2 itself, which treats the two instantons as distinct. The subtraction of 1 reflects that we need at least one world-sheet component with one boundary on the first instanton and one boundary on the second instanton. The factor of e^{2S_1} reflects that once we have one world-sheet with one boundary on the first instanton and one boundary on the second instanton, we can include any number of other world-sheet components whose boundaries lie wholly on the first instanton or wholly on the second instanton.

1/2 这个因子反映了一个事实: 两个全同瞬子的交换不会产生新贡献。这种对称性并没有被 S_2 本身体现，因为 S_2 将两个瞬子视作可区分的。减 1 反映了我们至少需要一个世界面分量，其一条边界在第一个瞬子上，另一条边界在第二个瞬子上。e^{2S_1} 因子反映了: 一旦我们存在一个这样的世界面，我们就可以添加任意数量其他世界面分量，这些分量的边界全部都在第一个瞬子上，或者全部都在第二个瞬子上。

It is now easy to verify using (613),(615), and (620) that S defined in (621) satisfies the BV master equation [227]:

现在利用 (613)、(615) 和 (620) 很容易验证，(621) 中定义的 S 满足 BV 主方程 [227]:

$$\frac{1}{2}\{S,S\}+\Delta S=\mathcal{O}\left(\mathcal{N}^3\right). \tag{622}$$

We expect that the order $\mathcal{N}^3$ terms are cancelled when including the effect of three instanton contribution.

我们预期，当计入三瞬子贡献后，$\mathcal{N}^3$ 阶项会被抵消。

Unitarity and Crossing Symmetry

幺正性与交叉对称性

In this subsection, we shall briefly describe how string field theory can be used to prove the unitarity and crossing symmetry of string amplitudes.

在本小节中，我们将简要介绍如何利用弦场论证明弦振幅的幺正性与交叉对称性。

Unitarity

幺正性

We can use string field theory to prove the unitarity of perturbative string amplitudes. The analysis can be divided into three parts:

我们可以利用弦场论证明微扰弦振幅的幺正性。该分析可分为三个部分:

1. First, we need to show that it is possible to impose appropriate reality condition on the string fields such that the string field theory action is real. A reality condition essentially relates the complex conjugate of the string fields to the original string field. It can take a different form for different components of the string field, e.g., $\psi_r^* = \psi_r$ or $\psi_r^* = -\psi_r$.

> 1. 首先，我们需要证明可以对弦场施加合适的实条件，使得弦场论作用量为实。实条件本质上是将弦场的复共轭与原弦场关联起来。它对弦场的不同分量可以有不同形式，例如 $\psi_r^* = \psi_r$ 或 $\psi_r^* = -\psi_r$ 。

2. We can then use the reality of the action to prove the Cutkosky rule that relates the anti-Hermitian part of an amplitude to the sum over all its cut diagrams. For $S = 1 + iT$, the cutting rule is the statement

> 2. 接下来我们可以利用作用量的实性证明 Cutkosky 规则，该规则将振幅的反厄米部分与所有切割图的求和关联起来。对于 $S = 1 + iT$ ，切割规则可表述为

$$T - T^\dagger = i \sum_n T^\dagger \, |n\rangle\langle n| \, T \;\Leftrightarrow\; \sum_n S^\dagger \, |n\rangle\langle n| \, S = 1, \tag{623}$$

where the sum over n runs over an orthonormal basis of states in the theory. This would be the statement of unitarity if the sum over n runs over only the physical states. The Cutkosky rules, however, are derived in the gauge fixed theory, where the sum also includes unphysical and pure gauge states. Therefore, (623) does not quite prove unitarity.

> 其中对 n 的求和遍历理论中的一组正交归一态基。如果求和仅对物理态进行，那么这就是幺正性的表述。但 Cutkosky 规则是在规范固定后的理论中推导出来的，求和中也包含了非物理态和纯规范态。因此，式 (623) 并未完全证明幺正性。

3. In order to address the issue mentioned above, we need to show that in (623), the contribution due to unphysical and pure gauge states in the sum over n cancels, and hence the sum over n , can be restricted to physical states only.

> 3. 为了解决上述问题，我们需要证明: 在式 (623) 中，对 n 求和里非物理态和纯规范态的贡献会相互抵消，因此对 n 的求和可以仅限制在物理态范围内。

Each of these steps can be carried out in string field theory, e.g., the proof of reality of the action may be found in [20], and the proof of decoupling of the unphysical and pure gauge states can be found in [228]. The main subtlety arises in the second step [229]. The usual proof of Cutkosky rules in perturbative quantum field theory uses the largest time equation [230] that uses some specific property of the position space Green's functions. The position space Green's functions, however, are not well defined in string theory because off-shell string amplitudes depend on external momenta by the exponential of a quadratic function of momenta-this can be seen, e.g., by noting that if the local coordinate w at a puncture at $w = 0$ is related to the coordinate z of the Riemann surface as $z = f(w)$, then the interaction vertex/off-shell amplitude contains a factor of $|f'(0)|^{k^2/2}$ since the conformal weight of the external off-shell vertex operator of momentum k has a contribution of the form $\left(\bar{h} + k^2/4, h + k^2/4\right)$ where $\bar{h}, h$ are k independent constants. Due to this exponential factor, the Fourier transform of the off-shell Green's function produces a highly non-local function in the position space and is hard to analyze. For this reason, we have to try to prove Cutkosky rules directly in momentum space.

以上每一步都可以在弦场论中完成，例如作用量实性的证明可见文献 [20]，非物理态和纯规范态退耦的证明可见文献 [228]。主要的难点出现在第二步 [229]。微扰量子场论中，Cutkosky 规则的常规证明使用了最大时间方程 [230]，该方程利用了位置空间格林函数的特定性质。但位置空间格林函数在弦论中并不是良定义的，因为 off-shell 弦振幅依赖于外动量，是动量二次型的指数函数——这一点可以通过下述例子看出: 如果孔点 $w = 0$ 处的局部坐标 w 与黎曼曲面的坐标 z 满足关系 $z = f(w)$，那么相互作用顶点/off-shell 振幅就会包含因子 $|f'(0)|^{k^2/2}$，因为动量为 k 的外 off-shell 顶点算符的共形权重有形式 $\left(\bar{h} + k^2/4, h + k^2/4\right)$ 的贡献，其中 $\bar{h}, h$ 是与 k 无关的常数。由于这个指数因子的存在，off-shell 格林函数的傅里叶变换在位置空间会得到一个高度非局域的函数，难以分析。因此我们必须尝试直接在动量空间证明 Cutkosky 规则。

Another difference with the usual quantum field theory lies in the choice of contour for integration over the internal loop energies. Due to the $|f'(0)|^{k^2/2}$ factors accompanying the vertices, we can ensure that the integrand of a Feynman diagram falls off rapidly for large positive k^2 by choosing $f'(0)$’s to be small. This ensures ultraviolet finiteness of the loop momentum integrals, provided we choose the contour of integration over each loop momentum k so that k^2 approaches ∞ as the contour approaches infinity. This in particular requires that the energy integrals run along the imaginary axis, at least for large energies. However, as explained below, for Lorentzian external momenta, it may not be possible to keep the energy integration contours fully along the imaginary axis. The strategy that one follows is the following [229]. We begin by taking the external momenta such that the spatial momenta are real and energies are purely imaginary. In this case, one can take the integration contours for loop energies fully along the imaginary axes without encountering any pole of the propagator, since all the internal energies are purely imaginary and hence the denominator factors in the propagators remain positive. We then rotate the phases of all the external energies simultaneously toward real values, so that the positive (negative) imaginary energies approach the positive (negative) real energies along the first (third) quadrant of the complex energy plane. Now, even if we take the internal loop energies (in some fixed labelling) to be all imaginary, due to energy conservation at the interaction vertices, some of the internal propagators will carry complex energy. Therefore, it is no longer guaranteed that the propagators will remain finite along the integration contour over loop energies running along the imaginary axes. Viewed in the complex loop energy plane, one can say that some of the poles of the integrand can approach the integration contour as we deform the external momenta. We deal with this situation by deforming the energy integration contours keeping their ends fixed at $\pm i\infty$ so as to avoid having the poles cross the integration contour. This has been shown in Fig. 22 with the crosses labeling the pole positions. It was shown in [229] that as long as the phases of all the external energies are kept the same, this can be done all the way to (almost) real values of the energies, and the resulting integrals can be taken to be the definition of the string amplitudes. This is the analog of the $i\varepsilon$ prescription for the propagators in usual quantum field theories. Furthermore, the amplitudes defined this way satisfy Cutkosky rules, establishing unitarity of string amplitudes to all orders in perturbation theory. [32]

与常规量子场论的另一处区别在于，我们对内部圈能量选择的积分围道不同。由于顶点伴随 $|f'(0)|^{k^2/2}$ 因子，我们可以通过选择 $f'(0)$ 取小值，确保费曼图的被积函数在大正 k^2 下快速衰减。只要我们选择每个圈动量 k 的积分围道，使得当围道趋向无穷时 k^2 趋向 ∞，就能保证圈动量积分是紫外有限的。这尤其要求能量积分沿虚轴进行，至少在大能量下是如此。但如下文所述，对于洛伦兹型外动量，能量积分围道未必能完全保持在虚轴上。我们采用的策略如下 [229]: 我们首先取外动量满足空间动量为实、能量为纯虚。在此情况下，我们可以将圈能量的积分围道完全沿虚轴选取，不会遇到传播子的任何极点，因为所有内部能量都是纯虚的，因此传播子的分母因子始终为正。随后我们将所有外能量的相位同时向实数值转动，使得正 (负) 虚能量沿复能量平面的第一 (第三) 象限趋向正 (负) 实能量。此时，即使我们将所有内部圈能量 (在某个固定标记下) 都取为虚数，由于相互作用顶点处的能量守恒，部分内部传播子会携带复能量。因此，我们无法再保证沿虚轴的圈能量积分围道上传播子始终有限。在复圈能量平面上看，可以说当我们形变外动量时，被积函数的部分极点会靠近积分围道。针对这种情况，我们通过形变能量积分围道、保持其端点固定在 $\pm i\infty$ 来避开极点 crossing 积分围道。图 22 展示了这一情况，叉号标记了极点位置。文献 [229] 已证明，只要所有外能量的相位保持一致，我们就可以一直这样操作直到能量达到 (几乎) 实数值，最终得到的积分可以作为弦振幅的定义。这就是常规量子场论中传播子 $i\epsilon$ 规则的类似推广。此外，按此方式定义的振幅满足 Cutkosky 规则，从而在微扰论的所有阶都证明了弦振幅的幺正性。[32]

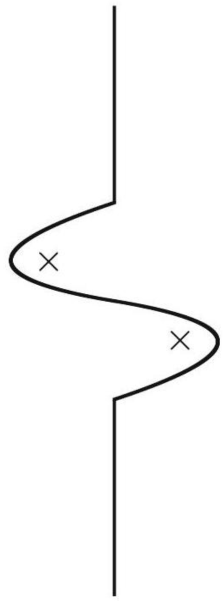

Fig. 22 This figure shows a typical integration contour over the energy flowing along an internal loop of the string field theory Feynman diagram. When the external momenta are purely imaginary, then we can take the internal energies also to be purely imaginary and the propagators have no poles. However, when the external momenta are deformed into the complex plane, the poles of the internal propagators could approach the imaginary axis of the internal energy. In that case, we deform the integration contour away from the poles, shown by $\times$, keeping its ends fixed at $\pm i\infty$ so that the integrand falls off at infinity

图 22 本图展示了弦场论费曼图内部流动能量的典型积分围道。当外能量为纯虚时，我们可以让内部能量也为纯虚，此时传播子没有极点。当外动量形变进入复平面后，内部传播子的极点会靠近内部能量的虚轴。此时我们将积分围道形变以避开极点，如图中 $\times$ 所示，保持围道端点固定在 $\pm i\infty$，从而保证被积函数在无穷远处衰减

Unitarity of the D-instanton contribution to string amplitudes can be proven in a similar way. Here, an additional subtlety arises in proving the reality of the action. Often in D-instanton systems, we encounter tachyonic open string modes, and as a result, the steepest descent contour along which we need to carry out the integration over the open string modes may lie along the imaginary axis, producing extra factors of i that can violate the reality of the D-instanton-induced effective action. In such cases, one either has to find a physical explanation for the lack of unitarity or find an appropriate prescription that restores reality of the effective action. An example of the first kind is provided by two dimensional non-critical bosonic string theory where the D-instanton-induced effective action has an imaginary part. Comparison of the results with that of a dual matrix model shows that the breakdown of unitarity can be attributed to the transition from the closed string states to a state of the system that does not have a closed string description [232]. This leads to an apparent lack of unitarity of the closed string amplitudes. An example of the second kind is provided by D-instanton anti-D-instanton contribution to the amplitudes in critical string theory. In this case, the exponential of the annulus partition function develops a pole when the separation between the instanton and the anti-instanton reaches a critical distance 2π , and in order to get a real contribution to the effective action, one needs to integrate over the relative separation using principle value prescription.

D 瞬子对弦振幅贡献的幺正性可以用类似方法证明。此处证明作用量的实性会多出一个难点。在 D 瞬子体系中，我们常会遇到快子开弦模，因此我们对开弦模积分所需的最速下降轮廓可能沿虚轴延伸，产生额外的 i 因子，这会破坏 D 瞬子诱导有效作用量的实性。在这种情况下，研究者要么需要为幺正性缺失找到物理解释，要么需要找到合适的规则恢复有效作用量的实性。二维非临界玻色弦理论提供了第一种情况的例子: 该理论中 D 瞬子诱导的有效作用量存在虚部。将结果与对偶矩阵模型的结果对比可知，幺正性破缺可归因于闭弦态向不存在闭弦描述的体系态的跃迁 [232]，这导致闭弦振幅看起来缺失幺正性。临界弦理论中 D 瞬子-反 D 瞬子对振幅的贡献提供了第二种情况的例子: 在此情形下，当瞬子与反瞬子的间距达到临界距离 2π 时，环面配分函数的指数会出现极点；而为了给有效作用量贡献实部，我们需要利用主值规则对相对间距积分。

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[32] The choice of integration contour described here can be shown to be equivalent to Witten’s $i\varepsilon$ prescription [39] for string amplitudes [231]. The latter prescription can be implemented directly at the level of the world-sheet expression for string amplitudes, but it is not clear how to prove Cutkosky rules directly in this formalism.

[32] 此处所述的积分轮廓选择可以被证明等价于弦振幅的 Witten $i\varepsilon$ 规则 [39][231]。后者可以直接在弦振幅的世界面表述层面实现，但目前尚不清楚如何在该形式体系中直接证明 Cutkosky 规则。

Crossing Symmetry

交叉对称性

Another desirable property of quantum field theory is crossing symmetry: the property that the S-matrix for the process $A+B \to C+D$ is related to that of the process $A+\bar{C} \to \bar{B}+D$ via suitable analytic continuation where bar denotes antiparticles. Note that both these S-matrices are given by the same Green’s function with different choice of external momenta. Thus, the key question is whether in the space of complex but on-shell external momenta, one can find a path that connects the two momentum configurations, with the Green's function remaining analytic all along this path.

量子场论的另一个理想性质是交叉对称性: 过程 $A+B\to C+D$ 的 S 矩阵与过程 $A+\bar{C}\to\bar{B}+D$ 的 S 矩阵可通过适当的解析延拓建立联系，其中上横线代表反粒子。注意这两个 S 矩阵都由同一个格林函数给出，仅外动量的选取不同。因此核心问题是: 在复在壳外动量空间中，是否存在一条连接两个动量构型的路径，使得格林函数沿整条路径保持解析。

In local quantum field theory, crossing symmetry was proved in [233,234] using an indirect method involving several steps.

定域量子场论中，交叉对称性已在文献 [233,234] 中通过分多步的间接方法证明。

1. One first proves that the off-shell Green's functions are analytic in a certain domain in the space of complex external momenta. This domain is known as the primitive domain of analyticity. The proof uses vanishing of commutators of local operators for space-like separated points.

1. 首先证明离壳格林函数在复外动量空间的某个特定区域内解析，该区域称为原始解析域。证明用到类空分离点处定域算符的对易子为零。

2. The primitive domain of analyticity can be shown to have no overlap with the domain in which the external momenta are on-shell. Therefore, it would seem that the analyticity of the off-shell Green's functions in the primitive domain of analyticity is of no use in the analysis of crossing symmetry, which is a property of the S-matrix. Nevertheless, using certain properties of functions of many complex variables, one can show that any function that is analytic in the primitive domain of analyticity is actually analytic in a bigger domain known as the envelope of holomorphy. The envelope of holomorphy does have overlap with domains in which the external momenta are on-shell.

2. 可以证明原始解析域与外动量为在壳的区域没有重叠。因此，原始解析域内离壳格林函数的解析性看似对分析交叉对称性 (S 矩阵的性质) 毫无用处。然而，利用多复变函数的特定性质可以证明: 任何在原始解析域内解析的函数，实际上在更大的区域——全纯包——内解析。全纯包确实与外动量为在壳的区域存在重叠。

3. One then shows that in the intersection of the envelop of holomorphy and the subspace of complex external momenta where all external momenta are on-shell, one can find a suitable path that connects the process $A+B\to C+D$ to the process $A+\bar{C}\to\bar{B}+D$.

3. 随后证明，在全纯包与所有外动量均为在壳的复外动量子空间的交集中，可以找到一条连接过程 $A+B\to C+D$ 和过程 $A+\bar{C}\to\bar{B}+D$ 的合适路径。

This QFT proof, however, required that the theory does not contain any massless states propagating in loops.

但该量子场论证明要求理论的圈传播中不包含任何零质量态。

As already discussed, the off-shell Green's functions in string field theory carry multiplicative factors of $\exp\left[-C_{ij}k_ik_j\right]$ for external momenta k_i and constants C_{ij} , and therefore do not have good locality property in the position space. As a result, the first step breaks down. Nevertheless, a perturbative proof of the holomorphy of the off-shell Green's function in the primitive domain of analyticity was given in [235] by explicit analysis of the string field theory Feynman diagrams and the prescription given in [229] for choosing the contours along which the internal energies are integrated. Once the first step of the proof is carried out, the second step goes through automatically since it relies only on general properties of functions of many complex variables, irrespective of how they arise. The third step can also be carried out following [234], since the only additional property of quantum field theory that the analysis of [234] uses is invariance of the off-shell Green's functions under Lorentz transformation with complex parameters. This is also valid for string theory amplitudes.

如前文所述，弦场论中的离壳格林函数带有外动量 k_i 的乘性因子 $\exp\left[-C_{ij}k_ik_j\right]$ 和常数 C_{ij} ，因此在位置空间没有良好的定域性。导致证明的第一步失效。尽管如此，文献 [235] 通过显式分析弦场论费曼图和 [229] 给出的内部能量积分 contour 选取规则，给出了离壳格林函数在原始解析域内全纯性的微扰证明。完成证明的第一步后，第二步会自动成立，因为它仅依赖多复变函数的一般性质，与这些函数的来源无关。第三步也可以沿用 [234] 的方法完成，因为 [234] 的分析仅额外要求离壳格林函数在复参数洛伦兹变换下不变，这一性质对弦理论振幅同样成立。

Note however that the proof assumes absence of massless states, which is not true in string theory. For this reason, the proof of crossing symmetry holds not for the full S-matrix elements but for a related quantity where we remove the contribution due to massless states from the internal propagators of the Feynman diagram. What this analysis establishes, however, is that the lack of crossing symmetry of string amplitudes has the same origin as the lack of crossing symmetry for quantum field theory amplitudes, namely, the presence of massless states. The fact that string (field) theory is apparently non-local does not pose any difficulty in the analysis of crossing symmetry, even though in the original QFT proof of crossing symmetry, locality played a crucial role in establishing the analyticity of the off-shell Green's function in the primitive domain of analyticity. Therefore, as far as crossing symmetry is concerned, string (field) theory behaves as a local quantum field theory.

但需要注意，该证明假设不存在零质量态，而这在弦理论中并不成立。因此，交叉对称性的证明并不适用于完整的 S 矩阵元，仅适用于从费曼图内部传播子中移除零质量态贡献后的相关量。但该分析表明，弦振幅不满足交叉对称性的起源，与量子场论振幅不满足交叉对称性的起源相同，即零质量态的存在。即便原始量子场论的交叉对称性证明中，定域性对建立离壳格林函数在原始解析域的解析性起到了关键作用，弦 (场) 论表观上的非定域性也不会给交叉对称性分析带来任何困难。因此，就交叉对称性而言，弦 (场) 论的行为与定域量子场论一致。

UV Finiteness

紫外有限性

String theory has been known to be UV finite since the early days of the theory, but string field theory offers a better perspective on the subject. For this let us review the usual heuristic argument for UV finiteness. By taking the limit where strings are point-like objects, we can collapse the world-sheet into world lines, and the integration over the moduli spaces of Riemann surfaces becomes integration over the lengths of the world-lines. This can be mapped to the conventional Feynman diagrams in quantum field theory by identifying the lengths of the world-lines as Schwinger parameters that are often used to rewrite the quantum field theory amplitudes. In this language, UV divergences appear from regions where a loop in the world-line diagram collapses to a point, i.e., all the length parameters along different propagators in the loop vanishes.

弦论自发展早期就被认为是紫外有限的，但弦场论为这一课题提供了更好的研究视角。下面我们先来回顾关于紫外有限性的常见启发式论证。当弦退化为类点对象时，世界面会坍缩为世界线，对黎曼曲面模空间的积分就转变为对世界线长度的积分。这可以对应到量子场论中的常规费曼图: 世界线的长度就是量子场论振幅改写中常用的施温格参数。在这套语言下，紫外发散来自世界线图中圈坍缩为点的区域，也就是圈中各传播子的长度参数都趋近于零的区域。

Given this, one can ask what this region would correspond to if we had fattened the world-lines into world-sheets. The answer is that such regions correspond to degenerate Riemann surfaces. On the other hand, IR divergences that come from the region where some of the world-line length parameters become large also correspond to degenerate Riemann surfaces. Since in string theory we are instructed to integrate over inequivalent Riemann surfaces, we should include the contribution from each degenerate Riemann surface only once. Using this, one can interpret all the UV divergent regions in the moduli spaces of Riemann surfaces as IR divergent regions. This shows that all divergences in string theory can be interpreted as IR divergences. Hence, string theory has no UV divergence. Such an interpretation is made manifest, for example, in the minimal area string diagrams discussed in section ”Minimal Area String Vertices: Witten Vertex and Closed String Polyhedra”. In this construction, all Riemann surfaces are built with propagators or cylinders of fixed finite circumference and string vertices that are nonsingular, as they contain no homotopically nontrivial closed curves of small length. All degenerations appear from infinitely long cylinders and thus correspond to the infrared configurations. In the representation where we express the string amplitudes as integrals over loop momenta, with infinite tower of states propagating along each propagator, ultraviolet finiteness can be seen by noting that each interaction vertex carries an exponential suppression factor when $\left(k^2 + m^2\right)$ of any external states becomes large. Therefore, neither the loop momentum integrals nor the sum over the infinite tower of propagating states encounters an ultraviolet divergence.

据此，我们可以提出一个问题: 如果我们将世界线加厚为世界面，这个区域会对应什么? 答案是，这类区域对应退化黎曼曲面。另一方面，部分世界线长度参数取大值的区域所产生的红外发散，同样对应退化黎曼曲面。由于弦论要求我们对不等价黎曼曲面积分，每个退化黎曼曲面的贡献应当仅被计入一次。据此，我们可以将黎曼曲面模空间中所有紫外发散区域解释为红外发散区域。这说明弦论中所有发散都可以被解读为红外发散，因此弦论不存在紫外发散。这种诠释在例如“极小面积弦顶点: 威滕顶点与闭弦多面体”一节讨论的极小面积弦图中体现得十分明显。在该构造中，所有黎曼曲面都由固定有限周长的传播子 (即柱面) 和非奇异弦顶点构造而成，因为其中不存在小长度的同伦非平凡闭曲线。所有退化都来自无穷长柱面，因此对应红外构型。在将弦振幅表示为对圈动量积分、且无穷多态塔沿每个传播子传播的表述中，可以观察到紫外有限性: 当任意外态的 $\left(k^2 + m^2\right)$ 变大时，每个相互作用顶点都携带一个指数抑制因子。因此，无论是圈动量积分还是对无穷传播态塔的求和，都不会出现紫外发散。

Now, one could argue that this is a matter of interpretation; what if we had reinterpreted all the divergences as UV divergences instead of IR divergences? The answer to this is that in quantum field theory, IR singularities of certain kind are necessary. For example, loop amplitudes are required to have certain poles and branch cuts associated with particle production in intermediate channels, and these arise precisely from the IR regions in a quantum field theory. Therefore, in order that string theory describes a good quantum theory, it must share the IR divergences that are present in quantum field theory. What we need to check is whether we have any additional divergences left after reproducing all the IR divergences that are present in quantum field theory. This is where string field theory proves useful by providing a fully consistent framework for analysis. Arguments presented in sections "Mass Renormalization and Vacuum Shift" and "Unitarity and Crossing Symmetry" show that divergences present in the theory are precisely those needed for a proper interpretation as a quantum theory. For example, if (623) failed to hold, we could assign the failure to a UV divergence by making a modular transformation that converts the IR region to the UV region in the world-sheet interpretation. However, (623) holds, and there is no leftover divergence that needs to be attributed to UV divergence in the theory.

有人可能会提出，这只是一个解释问题: 如果我们把所有发散都重新解读为紫外发散而非红外发散，结果会怎样？答案是: 在量子场论中，特定类型的红外奇点是必然存在的。例如，圈振幅必然存在与中间道粒子产生关联的极点和分支割，这些恰恰来自量子场论中的红外区域。因此，弦论要描述一个自洽的量子理论，就必须保留量子场论中已有的红外发散。我们需要验证的是，在重现量子场论所有红外发散后，是否还存在额外的发散。弦场论的用处就在于它为此分析提供了一套完全自洽的框架。“质量重整化与真空移位”和“幺正性与交叉对称性”两节给出的论证表明，该理论中存在的发散恰好是量子理论合理解释所必需的。例如，如果式 (623) 不成立，我们就可以通过模变换在世界面诠释中将红外区域转换为紫外区域，将该不成立归因于紫外发散。但实际上式 (623) 成立，理论中不存在需要归为紫外发散的剩余发散。

To summarize, the fact that string field theory allows us to express the amplitudes as a sum over Feynman diagrams whose infrared properties are the same as those of a quantum field theory and that the momentum integrals (and sum over infinite number of internal states) are finite due to the exponential suppression of the interaction vertices discussed in section "Unitarity and Crossing Symmetry" shows the absence of UV divergences in the theory.

综上，弦场论让我们可以将振幅表示为费曼图的和，这些费曼图的红外性质与量子场论一致，且由于“幺正性与交叉对称性”一节讨论的相互作用顶点的指数压制，动量积分 (以及对无穷多内态的求和) 都是有限的，这一事实证明该理论不存在紫外发散。

Some Future Directions

若干未来研究方向

At the time of this writing (2024), covariant string field theory has been developed and studied for about forty years. As we have seen, there are formulations of all the string field theories, but the search continues for more efficient formulations.

截至本文撰写时 (2024 年)，协变弦论已经发展和研究了约四十年。正如我们所见，目前所有弦场论都已有对应的表述，但对更高效表述的探索仍在继续。

There are also some obvious outstanding questions that could perhaps be tackled in the context of the present formulations of string field theory. Finally, there are a number of physical questions whose answers would be of current interest, and dealing with them would surely better our understanding of string field theory. We briefly consider some of these directions.

此外还有一些明显悬而未决的问题，或许可以在现有弦场论表述的框架下解决。最后，还有诸多当下值得研究的物理问题，解答这些问题必定会增进我们对弦场论的理解。我们将简要探讨其中部分研究方向。

1. It can be argued that just as bosonic string field theories are nicely formulated in terms of moduli spaces of Riemann surfaces, superstring field theories would have a more foundational definition if built using moduli spaces of super-Riemann surfaces. In such formulation, perhaps the present constructions using PCOs would be seen to arise naturally by making choices in the description of such supermoduli spaces. Early relevant work on supermoduli spaces was done by Belopolsky [236,237]. More recently, this subject has been discussed at length by Witten [201], and the equivalence between the supermoduli and the PCO formalisms at the level of the world-sheet theory was established by Wang and Yin [238, 239]. For some steps in the construction of the NS sector of the open superstring field theory using supermoduli space, see [149], with the extension to the R sector considered in [240].

1. 有观点认为，正如玻色弦场论可以通过黎曼曲面模空间得到优美表述，超弦场论若基于超黎曼曲面模空间构建，也能获得更具基础意义的定义。在这类表述中，目前使用 PCO 的构造或许可以自然地通过对超模空间描述的选择导出。Belopolsky 早已对超模空间开展了相关研究 [236,237]。近年来 Witten 对该主题进行了详尽讨论 [201]，Wang 与 Yin 证明了世界面理论层面超模空间形式体系与 PCO 形式体系等价 [238,239]。关于利用超模空间构造开超弦场论 NS 区的相关进展，参见文献 [149]，拓展至 R 区的研究参见 [240]。

2. A possibility is that a supermoduli space formulation would yield canonical insertions of PCOs. This would be highly desirable, as it could help the construction of string field theory solutions, which nowadays only exist for the canonically associative open bosonic string field theory and open super-string field theory in the large Hilbert space. One option was discussed in section ”String Vertices for Open Superstring Field Theory”, where the insertions occur on the boundary of the coordinate disks associated with the punctures. A canonical distribution of PCOs on the bulk of Riemann surfaces is not yet available for the version of heterotic and type II string theories of [19, 20] .

2. 一种可能性是超模空间表述可以给出 PCO 的正则插入。这一点非常值得期待，因为它有助于构造弦场论解——目前仅存在于大希尔伯特空间下的正则结合开玻色弦场论与开超弦场论中。在“开超弦场论的弦顶点”一节已经讨论过一种方案，该方案中 PCO 插入在与刺点对应的坐标圆盘边界上。对于 [19, 20] 的杂化弦与 II 型弦理论，目前还没有得到黎曼曲面体上 PCO 的正则分布。

3. String field theories reviewed in this chapter are all formulated given a string background, essentially a CFT (or BCFT) that is coupled to ghost sectors. CFT defines the consistent background, which happens to be a solution represented by the zero string field. Ideally, one would like to formulate the theory around arbitrary backgrounds, not just consistent backgrounds, just like Einstein’s equations are written for arbitrary metrics on a manifold. Background independent formulations were considered in [165, 166] using the BV approach. An approach based on L_∞ algebras with a product without input (often called a ”curved” algebra) was explored in [100, 241] . For open strings, the possibility of using a special background to formulate the theory seemed promising and gave rise to ”vacuum string field theory,” in which the tachyon vacuum, a background with no open string degrees of freedom, was used as a starting point [242-245].

3. 本章回顾的弦场论都是在给定弦背景下构造的，该背景本质上是一个耦合了鬼场部分的共形场论 (CFT，或边界共形场论 BCFT)。共形场论定义了一致背景，对应零弦场表示的解。理想情况下，我们希望理论能够围绕任意背景表述，而非仅局限于一致背景——就像爱因斯坦方程可以对流形上的任意度规写出。已有研究利用 BV 方法讨论了背景无关表述 [165,166]。[100, 241] 中探讨了一种基于带无输入乘积 (常称为“弯曲”代数) 的 L_∞ 代数的方案。对于开弦，利用特殊背景构造理论的思路看起来很有前景，由此诞生了“真空弦场论”，该理论以快子真空 (一个没有开弦自由度的背景) 作为出发点 [242-245]。

4. The simple structure of the cubic bosonic open string field theory has allowed the construction of a good number of classical solutions, including time-dependent ones. The relative simplicity of the associative star product allowed for this development. To date, there is no known exact classical solution of bosonic closed string field theory, nor has the tachyon vacuum for this theory been convincingly identified [63, 126]. It seems quite plausible that the simplest vertices as far as finding solutions are the ”polyhedral” vertices arising from Strebel quadratic differentials (or minimal area metrics). After all, these vertices are the natural generalization of the open string associative vertex to closed strings. Finding exact solutions of classical closed string field theory would be a welcome breakthrough.

4. 三次开玻色弦场论的简单结构已经允许人们构造出大量经典解，其中也包括含时间依赖的解，结合乘积相对简便的性质为这一发展提供了便利。迄今为止，人们既没有找到玻色闭弦场论的已知精确经典解，也没能令人信服地确定该理论的快子真空 [63, 126]。从求解的角度看，最简单的顶点很可能是来自 Strebel 二次微分 (或极小面积度量) 的“多面体”顶点。毕竟这些顶点本身就是开弦结合顶点向闭弦的自然推广。找到玻色闭弦场论的经典精确解会是一项值得期待的突破。

Some recent work pointing in this direction include trying to make open string field theory analogous to closed string field theory by adding stubs to open string vertices [246, 247] and formulating the non-polynomial stubbed open string field theory as a cubic theory with the use of auxiliary fields [248, 249]. There is also insight to be gained by studying QFT with stubs [250] and, for the problem of closed string tachyon condensation, by investigating how stubs affect effective potentials in field theory [251]. On a different vein, there are investigations of closed string field theory aiming to relax the level matching constraint $L_0^- = 0$ [252, 253].

近年来沿这个方向的研究包括: 通过给开弦顶点添加桩，尝试让开弦场论具备与闭弦场论类似的结构 [246, 247]，以及利用辅助场将非多项式带桩开弦场论表述为三次理论 [248, 249]。研究带桩量子场论也能带来新启发 [250]，针对闭弦快子凝聚问题，探究桩对场论中有效势的影响也能获得收获 [251]。除此之外，目前也有放松能级匹配约束 $L_0^- = 0$ 的闭弦场论研究 [252, 253]。

5. Among the classical solutions of closed bosonic string field theory, or heterotic or closed superstring field theory, black hole solutions would be among the most interesting ones. One of the outstanding issues involving black hole solutions in string theory is: Given a black hole solution, described by a two-dimensional world-sheet conformal field theory, how can we calculate its entropy? In other words, what is the analog of Wald’s formula [254], describing the entropy of a black hole in any higher derivative theory of gravity, in string theory? Classical string field theory may be able to answer this question, particularly since in the Gibbons-Hawking formalism [255], the computation of the entropy can be reformulated as the computation of the on-shell action as a function of the period of the Euclidean time circle. In this context, the recent result by Erler [256] showing that the vanishing of the bulk terms in the on-shell action is a consequence of the dilaton

theorem could provide a clue. A somewhat different approach to this problem has been suggested in [257, 258].

5. 在闭玻色弦场论、杂化弦场论或闭超弦场论的经典解中，黑洞解会是最有意思的解之一。弦论中涉及黑洞解的一个悬而未决的问题是: 对于一个由二维世界面共形场论描述的黑洞解，我们如何计算它的熵？换句话说，弦论中，描述任意高阶导数引力理论中黑洞熵的瓦尔德公式 [254] 的对应形式是什么？经典弦场论或许能够解答这个问题，尤其是在吉本斯-霍金形式体系 [255] 中，熵的计算可以重新表述为欧几里得时间圆周周期函数的在壳作用量计算。在此背景下，Erler 最近的结果 [256] 表明，在壳作用量中体项为零是胀子定理的推论，这或许能提供一条线索。该问题的另一种不同思路已在文献 [257, 258] 中提出。

6. The dilaton theorem, reviewed in section ”Dilaton Theorem” for the case of closed bosonic string field theory, has not yet been extended to open-closed theories and heterotic and type II closed strings. There is also the issue of constant or Gibbons-Hawking-York type terms that could be present in the closed string field theory action when dealing with noncompact spaces or on spaces with boundaries [259, 260]. The constant terms in the action (cosmological terms) in open-closed string field theory have been discussed in [106], relating them to world-sheet partition functions on the disk.

6. 本文“胀子定理”一节回顾了闭玻色弦场论情形下的胀子定理，该定理尚未推广到开-闭弦理论、杂化弦和 II 型闭弦。此外，处理非紧致空间或带边界空间时，闭弦场论作用量中还可能存在常数项或吉本斯-霍金-约克型项的问题 [259, 260]。开-闭弦场论作用量中的常数项 (宇宙学项) 已在文献 [106] 中讨论，作者将其与圆盘上的世界面配分函数联系了起来。

7. It remains to be seen if string field theory can provide insights into holography, perhaps a ”proof” of the AdS/CFT correspondence, the central example of an open-closed duality, a duality between large N gauge theories and closed strings. Some directions based on open string field theory, A_∞ and L_∞ algebras, are explored in [72, 261, 262] . There is also work using open-closed string field theory in the background of N extended D-branes, with N large [105,112]. Hyperbolic open-closed vertices [135] are relevant here. Some earlier research considered simpler versions of the open-closed duality, such as the duality between Chern-Simons and the closed topological A-model [263]. In the framework of string field theory, [264] studied the duality between topological matrix models and open string field theory on N extended Liouville branes.

7. 弦场论能否为全息原理提供洞见，或许能给出 AdS/CFT 对 (大 N 规范理论与闭弦之间的对偶，是开-闭对偶的核心例子) 的“证明”，这一点仍有待观察。基于开弦场论、A_∞ 和 L_∞ 代数的若干研究方向已在 [72, 261, 262] 中探讨。也已有研究在 N 大的背景下，利用 N 扩展 D 膜背景下的开-闭弦场论开展工作，双曲开-闭顶点 [135] 与此相关。部分早期研究考察了开-闭对偶的更简单形式，例如陈-西蒙斯理论与闭拓扑 A 模型之间的对偶 [263]。在弦场论框架内，文献 [264] 研究了拓扑矩阵模型与 N 扩展刘维尔膜上开弦场论之间的对偶。

In the special cases where the background contains pure NSNS flux [265-270], we have an exact CFT description of the world-sheet theory, and it is possible to formulate string field theory in this background following the procedure described in this chapter. We can then try to identify what computations in this string field theory will give the boundary S-matrix in AdS and try to relate it to the correlations functions of local operators in the boundary theory.

在背景仅含 NSNS 通量的特殊情形下 [265-270]，我们拥有世界面理论的精确 CFT 描述，因此可以遵循本章介绍的步骤在该背景下构造弦场论。之后我们可以尝试找出，该弦场论中的哪些计算会给出 AdS 中的边界 S 矩阵，并尝试将其与边界理论中局域算符的关联函数联系起来。

8. It may be possible to use string field theory to address some of the perturbative results relevant for string phenomenology, e.g., computation of the Kahler potential in $\mathcal{N} = 1$ supersymmetric string compactification. We have already mentioned the use of string field theory for the computation of D-instanton correction to the superpotential. Another potential application of string field theory will be in developing a systematic procedure for computing amplitudes in the presence of RR flux. See [209] for some recent progress in this direction.

8. 弦场论或许可以用来处理与弦唯象学相关的若干微扰结果，例如计算 $\mathcal{N} = 1$ 超对称弦紧致化中的凯勒势。我们已经提到过，弦场论可用于计算超势的 D 瞬子修正。弦场论的另一个潜在应用是发展出一套系统步骤，用来计算 RR 通量存在时的振幅。该方向的最新进展可参见文献 [209]。

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References

参考文献

1. W. Siegel, Covariantly Second Quantized String. 2. Phys. Lett. B 149, 157 (1984). https://doi.org/10.1016/0370-2693(85)91661-2

2. M. Kato, K. Ogawa, Covariant quantization of string based on BRS invariance. Nucl. Phys. B 212, 443 (1983)

3. M. Kaku, K. Kikkawa, The field theory of relativistic strings. I. Trees. Phys. Rev. D 10 (1974), 1110. https://doi.org/10.1103/PhysRevD.10.1110

4. S. Mandelstam, The interacting string picture and functional integration, in Unified String Theories, ed. by M. Green, D. Gross (World Scientific, Singapore, 1986)

5. E. Witten, Noncommutative geometry and string field theory. Nucl. Phys. B 268, 253-294 (1986). https://doi.org/10.1016/0550-3213(86)90155-0

6. B. Zwiebach, Closed string field theory: quantum action and the B-V master equation. Nucl. Phys. B 390, 33-152 (1993). https://doi.org/10.1016/0550-3213(93)90388-6 [arXiv:hep-th/9206084 [hep-th]].

7. B. Zwiebach, Quantum open string theory with manifest closed string factorization. Phys. Lett. B 256, 22-29 (1991). https://doi.org/10.1016/0370-2693(91)90212-9

8. B. Zwiebach, Interpolating string field theories. Mod. Phys. Lett. A 7, 1079-1090 (1992). https://doi.org/10.1142/S02177 [arXiv:hep-th/9202015 [hep-th]]

9. B. Zwiebach, Oriented open-closed string theory revisited. Annals Phys. 267, 193-248 (1998). https://doi.org/10.1006/ap [arXiv:hep-th/9705241 [hep-th]]

10. H. Erbin, String field theory: a modern introduction. Lect. Notes Phys. 980, 1-421 (2021). ISBN 978-3-030-65320-0, 978-3-030-65321-7. https://doi.org/10.1007/978-3-030-65321-7 [arXiv:2301.01686 [hep-th]]

11. C. Maccaferri, String Field Theory [arXiv:2308.00875 [hep-th]]

12. T. Erler, Four lectures on analytic solutions in open string field theory. Phys. Rept. 980, 1-95 (2022). https://doi.org/10.1016/j.physrep.2022.06.004 [arXiv:1912.00521 [hep-th]]

13. T. Erler, Four lectures on closed string field theory. Phys. Rept. 851, 1-36 (2020). https://doi.org/10.1016/j.physrep.2020 [arXiv:1905.06785 [hep-th]]

14. M. Kudrna, Level Truncation Approach to Open String Field Theory [arXiv:2101.07678 [hep-th]]

15. C. de Lacroix, H. Erbin, S. P. Kashyap, A. Sen, M. Verma, Closed superstring field theory and its applications. Int. J. Mod. Phys. A 32(28n29), 1730021 (2017). https://doi.org/10.1142/ S0217751X17300216 [arXiv:1703.06410 [hep-th]]

16. Y. Okawa, Analytic methods in open string field theory. Prog. Theor. Phys. 128, 1001-1060 (2012). https://doi.org/10.1143/PTP.128.1001

17. E. Fuchs, M. Kroyter, Analytical solutions of open string field theory. Phys. Rept. 502 (2011), 89-149. https://doi.org/10.1016/j.physrep.2011.01.003 [arXiv:0807.4722 [hep-th]]

18. W. Taylor, B. Zwiebach, D-branes, tachyons, and string field theory. https://doi.org/10.1142/ 9789812702821_0012 [arXiv:hep-th/0311017 [hep-th]]

19. A. Sen, BV master action for heterotic and Type II string field theories. JHEP 02, 087 (2016). https://doi.org/10.1007/JH [arXiv:1508.05387 [hep-th]]

20. A. Sen, Reality of superstring field theory action. JHEP 11, 014 (2016). https://doi.org/10.1007/JHEP11(2016)014 [arXiv:1606.03455 [hep-th]]

21. S. Faroogh Moosavian, A. Sen, M. Verma, Superstring field theory with open and closed strings. JHEP 01, 183 (2020). https://doi.org/10.1007/JHEP01(2020)183 [arXiv:1907.10632 [hep-th]]

22. A. Sen, B. Zwiebach, On the normalization of open-closed string amplitudes [arXiv:2405.03784 [hep-th]]

23. N. Berkovits, SuperPoincare invariant superstring field theory. Nucl. Phys. B 450, 90- 102 (1995).

[erratum: Nucl. Phys. B 459, 439-451 (1996)]. https://doi.org/10.1016/0550- 3213(95)00259-U [arXiv:hep-th/9503099 [hep-th]]

24. N. Berkovits, A New approach to superstring field theory. Fortsch. Phys. 48, 31-36 (2000) [arXiv:hep-th/9912121 [hep-th]]

25. Y. Okawa, B. Zwiebach, Heterotic string field theory. JHEP 07, 042 (2004). https://doi.org/ 10.1088/1126-6708/2004/07/042 [arXiv:hep-th/0406212 [hep-th]]

26. N. Berkovits, Y. Okawa, B. Zwiebach, WZW-like action for heterotic string field theory. JHEP 11, 038 (2004). https://doi.org/10.1088/1126-6708/2004/11/038 [arXiv:hep-th/0409018 [hep-th]]

27. H. Matsunaga, Nonlinear gauge invariance and WZW-like action for NS-NS superstring field theory. JHEP 09, 011 (2015). https://doi.org/10.1007/JHEP09(2015)011 [arXiv:1407.8485 [hep-th]]

28. T. Erler, S. Konopka, I. Sachs, NS-NS sector of closed superstring field theory. JHEP 08, 158 (2014). https://doi.org/10.1007/JHEP08(2014)158 [arXiv:1403.0940 [hep-th]]

29. H. Kunitomo, Y. Okawa, Complete action for open superstring field theory. PTEP 2016(2), 023B01 (2016). https://doi.org/10.1093/ptep/ptv189 [arXiv:1508.00366 [hep-th]]

30. H. Kunitomo, Y. Okawa, H. Sukeno, T. Takezaki, Fermion scattering amplitudes from gauge-invariant actions for open superstring field theory [arXiv:1612.00777 [hep-th]]

31. D. Friedan, S.H. Shenker, E.J. Martinec, Covariant quantization of superstrings. Phys. Lett. B 160, 55-61 (1985). https://doi.org/10.1016/0370-2693(85)91466-2

32. G. Segal, The definition of conformal field theory, in Topology, Geometry and Quantum Field Theory: Proceedings of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal. London Mathematical Society Lecture Note Series (Cambridge University, Cambridge, 2004), pp. 421-422

33. L. Alvarez-Gaume, C. Gomez, G.W. Moore, C. Vafa, Strings in the operator formalism. Nucl. Phys. B 303, 455-521 (1988). https://doi.org/10.1016/0550-3213(88)90391-4

34. K. Costello, B. Zwiebach, Hyperbolic string vertices. JHEP 02, 002 (2022). https://doi.org/ 10.1007/JHEP02(2022)002 [arXiv:1909.00033 [hep-th]]

35. A. LeClair, M.E. Peskin, C.R. Preitschopf, String field theory on the conformal plane. 1. Kinematical Principles. Nucl. Phys. B 317, 411-463 (1989). https://doi.org/10.1016/0550- 3213(89)90075-8

36. A. LeClair, M.E. Peskin, C.R. Preitschopf, Nucl. Phys. B 317, 464-508 (1989). https://doi.org/10.1016/0550-3213(89)90076-X

37. A. Belopolsky, B. Zwiebach, Off-shell closed string amplitudes: towards a computation of the tachyon potential. Nucl. Phys. B 442, 494-532 (1995). https://doi.org/10.1016/0550- 3213(95)00094-9 [arXiv:hep-th/9409015 [hep-th]]

38. M. Schiffer, D.C. Spencer, Functionals on Finite Riemann Surfaces (Princeton University Press, Princeton, 1954); F.P. Gardiner, Duke. Math. J. 42, 371 (1975); S. Nag, Proc. Indian Acad. Sci. (Math. Sci.) 94, 111 (1985)

39. E. Witten, The Feynman $i\varepsilon$ in string theory. JHEP 04,055 (2015). https://doi.org/10.1007/ JHEP04(2015)055 [arXiv:1307.5124 [hep-th]]

40. L. Eberhardt, S. Mizera, Evaluating one-loop string amplitudes. SciPost Phys. 15(3), 119 (2023). https://doi.org/10.21468/SciPostPhys.15.3.119 [arXiv:2302.12733 [hep-th]]

41. E.P. Verlinde, H.L. Verlinde, Multiloop calculations in covariant superstring theory. Phys. Lett. B 192, 95-102 (1987). https://doi.org/10.1016/0370-2693(87)91148-8

42. A. Sen, Off-shell amplitudes in superstring theory. Fortsch. Phys. 63, 149-188 (2015). https:// doi.org/10.1002/prop.201500002 [arXiv:1408.0571 [hep-th]]

43. A. Sen, E. Witten, Filling the gaps with PCO's. JHEP 09, 004 (2015). https://doi.org/10.1007/ JHEP09(2015)004 [arXiv:1504.00609 [hep-th]]

44. I.A. Batalin, G.A. Vilkovisky, Gauge algebra and quantization. Phys. Lett. 102B, 27 (1981). https://doi.org/10.1016/0370 2693(81)90205-7

45. I.A. Batalin, G.A. Vilkovisky, Quantization of gauge theories with linearly dependent generators. Phys. Rev. D 28, 2567 (1983). Erratum: [Phys. Rev. D 30, 508 (1984)]. https:// doi.org/10.1103/PhysRevD.28.2567. https://doi.org/10.1103/PhysRevD.30.508

46. M. Henneaux, C. Teitelboim, Quantization of Gauge Systems (Princeton University Press, Princeton, 1992)

47. A.S. Schwarz, Geometry of Batalin-Vilkovisky quantization. Commun. Math. Phys. 155, 249-260 (1993). https://doi.org/10.1007/BF02097392 [arXiv:hep-th/9205088 [hep-th]]

48. W. Siegel, B. Zwiebach, Gauge string fields. Nucl. Phys. B 263, 105-128 (1986). https://doi.org/10.1016/0550-3213(86)90030-1

49. T. Banks, M.E. Peskin, Gauge invariance of string fields. Nucl. Phys. B 264, 513-547 (1986). https://doi.org/10.1016/055 3213(86)90496-7

50. A. Neveu, H. Nicolai, P.C. West, New symmetries and ghost structure of covariant string theories. Phys. Lett. B 167, 307-314 (1986). https://doi.org/10.1016/0370-2693(86)90351-5

51. A. Neveu, P.C. West, The interacting gauge covariant bosonic string. Phys. Lett. B 168, 192- 200 (1986). https://doi.org/10.1016/0370-2693(86)90962-7

52. H. Hata, K. Itoh, T. Kugo, H. Kunitomo, K. Ogawa, Manifestly covariant field theory of interacting string. Phys. Lett. B 172, 186-194 (1986). https://doi.org/10.1016/0370- 2693(86)90834-8

53. H. Hata, K. Itoh, T. Kugo, H. Kunitomo, K. Ogawa, Manifestly covariant field theory of interacting string. 2. Phys. Lett. B 172, 195-203 (1986). https://doi.org/10.1016/0370- 2693(86)90835-X

54. H. Hata, K. Itoh, T. Kugo, H. Kunitomo, K. Ogawa, Pregeometrical string field theory: creation of space-time and motion. Phys. Lett. B 175, 138-144 (1986). https://doi.org/10.1016/0370-2693(86)90703-3

55. T. Kugo, B. Zwiebach, Target space duality as a symmetry of string field theory. Prog. Theor. Phys. 87, 801-860 (1992). https://doi.org/10.1143/ptp/87.4.801 [arXiv:hep-th/9201040 [hep-th]]

56. M.R. Gaberdiel, B. Zwiebach, Tensor constructions of open string theories. 1: Foundations. Nucl. Phys. B 505, 569-624 (1997). https://doi.org/10.1016/S0550-3213(97)00580-4 [arXiv:hep-th/9705038 [hep-th]]

57. A. Sen, Universality of the tachyon potential. JHEP 12, 027 (1999). https://doi.org/10.1088/ 1126-6708/1999/12/027 [arXiv:hep-th/9911116 [hep-th]]

58. M. Cho, B. Mazel, X. Yin, Rolling tachyon and the Phase Space of Open String Field Theory [arXiv:2310.17895 [hep-th]]

59. T. Erler, S. Konopka, I. Sachs, Resolving Witten's superstring field theory. JHEP 04, 150 (2014). https://doi.org/10.1007/JHEP04(2014)150 [arXiv:1312.2948 [hep-th]]

60. T. Erler, S. Konopka, I. Sachs, Ramond equations of motion in superstring field theory. JHEP 11, 199 (2015). https://doi.org/10.1007/JHEP11(2015)199 [arXiv:1506.05774 [hep-th]]

61. C.B. Thorn, String field theory. Phys. Rept. 175, 1-101 (1989). https://doi.org/10.1016/0370- 1573(89)90015-X

62. I. Ellwood, J. Shelton, W. Taylor, Tadpoles and closed string backgrounds in open string field theory. JHEP 07, 059 (2003). https://doi.org/10.1088/1126-6708/2003/07/059 [arXiv:hep-th/0304259 [hep-th]]

63. H. Yang, B. Zwiebach, A Closed string tachyon vacuum?. JHEP 09, 054 (2005). https://doi.org/10.1088/1126-6708/2005/09/054 [arXiv:hep-th/0506077 [hep-th]]

64. C. Hull, B. Zwiebach, Double field theory. JHEP 09, 099 (2009). https://doi.org/10.1088/ 1126-6708/2009/09/099 [arXiv:0904.4664 [hep-th]]

65. H. Sonoda, Sewing conformal field theories. Nucl. Phys. B 311, 401-416 (1988). https://doi.org/10.1016/0550-3213(88)90066

66. H. Sonoda, Sewing conformal field theories. 2. Nucl. Phys. B 311, 417-432 (1988). https:// doi.org/10.1016/0550-3213(88)90067-3

67. S.B. Giddings, E.J. Martinec, E. Witten, Modular invariance in string field theory. Phys. Lett. B 176, 362-368 (1986). https://doi.org/10.1016/0370-2693(86)90179-6

68. B. Zwiebach, A Proof that Witten's open string theory gives a single cover of moduli space. Commun. Math. Phys. 142, 193-216 (1991). https://doi.org/10.1007/BF02099176

69. A. Sen, Wilsonian effective action of superstring theory. JHEP 01, 108 (2017). https://doi.org/10.1007/JHEP01(2017)10 [arXiv:1609.00459 [hep-th]]

70. H. Kajiura, Noncommutative homotopy algebras associated with open strings. Rev. Math. Phys. 19, 1-99 (2007). https://doi.org/10.1142/S0129055X07002912 [arXiv:math/0306332 [math.QA]]

71. H. Erbin, C. Maccaferri, M. Schnabl, J. Vošmera, Classical algebraic structures in string theory effective actions. JHEP 11, 123 (2020). https://doi.org/10.1007/JHEP11(2020)123 [arXiv:2006.16270 [hep-th]]

72. D. Koyama, Y. Okawa, N. Suzuki, Gauge-invariant operators of open bosonic string field theory in the low-energy limit [arXiv:2006.16710 [hep-th]]

73. A.S. Arvanitakis, O. Hohm, C. Hull, V. Lekeu, Homotopy transfer and effective field theory I: Tree-level [arXiv:2007.07942 [hep-th]]

74. R.K. Singh, Algebraic Structures in Closed Superstring Field Theory, Homotopy Transfer and Effective Actions [arXiv:2405.08063 [hep-th]]

75. C. Maccaferri, J. Vošmera, Closed string deformations in open string field theory. Part I. Bosonic string. JHEP 09, 047 (2021). https://doi.org/10.1007/JHEP09(2021)047

76. T. Erler, H. Matsunaga, Mapping between Witten and lightcone string field theories. JHEP 11, 208 (2021). https://doi.org/10.1007/JHEP11(2021)208 [arXiv:2012.09521 [hep-th]]

77. A. Sen, Gauge invariant 1PI effective action for superstring field theory. JHEP 06, 022 (2015). https://doi.org/10.1007/JHEP06(2015)022 [arXiv:1411.7478 [hep-th]]

78. H. Hata, B. Zwiebach, Developing the covariant Batalin-Vilkovisky approach to string theory. Annals Phys. 229, 177-216 (1994). https://doi.org/10.1006/aphy.1994.1006 [arXiv:hep-th/9301097 [hep-th]]

79. K.J. Costello, The Gromov-Witten potential associated to a TCFT [arXiv:math/0509264 [math.QA]]

80. A. Sen, B. Zwiebach, A Proof of local background independence of classical closed string field theory. Nucl. Phys. B 414, 649-714 (1994). https://doi.org/10.1016/0550- 3213(94)90258-5 [arXiv:hep-th/9307088 [hep-th]]

81. A. Sen, On the background independence of string field theory. Nucl. Phys. B 345, 551-583 (1990). https://doi.org/10.1016/0550-3213(90)90400-8

82. K. Ranganathan, Nearby CFTs in the operator formalism: the Role of a connection. Nucl. Phys. B 408, 180-206 (1993). https://doi.org/10.1016/0550-3213(93)90136-D [arXiv:hep-th/9210090 [hep-th]]

83. K. Ranganathan, H. Sonoda, B. Zwiebach, Connections on the state space over conformal field theories. Nucl. Phys. B 414, 405-460 (1994). https://doi.org/10.1016/ 0550-3213(94)90436-7 [arXiv:hep-th/9304053 [hep-th]]

84. A. Sen, B. Zwiebach, Quantum background independence of closed string field theory. Nucl. Phys. B 423, 580-630 (1994). https://doi.org/10.1016/0550-3213(94)90145-7 [arXiv:hep-th/9311009 [hep-th]]

85. A. Sen, Background independence of closed superstring field theory. JHEP 02, 155 (2018). https://doi.org/10.1007/JHE [arXiv:1711.08468 [hep-th]]

86. S. Mukherji, A. Sen, Some all order classical solutions in nonpolynomial closed string field theory. Nucl. Phys. B 363, 639-664 (1991). https://doi.org/10.1016/0550-3213(91)80037-M

87. B. Mazel, J. Sandor, C. Wang, X. Yin, Conformal Perturbation Theory and Tachyon-Dilaton Eschatology via String Fields [arXiv:2403.14544 [hep-th]]

88. O. Bergman, B. Zwiebach, The Dilaton theorem and closed string backgrounds. Nucl. Phys. B 441, 76-118 (1995). https://doi.org/10.1016/0550-3213(95)00022-K [arXiv:hep-th/9411047 [hep-th]].

89. S. Rahman, B. Zwiebach, Vacuum vertices and the ghost dilaton. Nucl. Phys. B 471, 233-245 (1996). https://doi.org/10.1016/0550-3213(96)00179-4 [arXiv:hep-th/9507038 [hep-th]]

90. J. Distler, P.C. Nelson, Topological couplings and contact terms in 2-d field theory. Commun. Math. Phys. 138, 273-290 (1991). https://doi.org/10.1007/BF02099493

91. J. Polchinski, Factorization of bosonic string amplitudes. Nucl. Phys. B 307, 61-92 (1988). https://doi.org/10.1016/0550 3213(88)90522-6

92. O. Hohm, B. Zwiebach, L_∞ algebras and field theory. Fortsch. Phys. 65(3-4),1700014 (2017). https://doi.org/10.1002/pr [arXiv:1701.08824 [hep-th]]

93. R. Blumenhagen, M. Fuchs, M. Traube, $\mathcal{W}$ algebras are L_∞ algebras. JHEP 07,060 (2017). https://doi.org/10.1007/JHEF [arXiv:1705.00736 [hep-th]]

94. L. Borsten, H. Kim, B. Jurčo, T. Macrelli, C. Saemann, M. Wolf, Double copy from homotopy algebras. Fortsch. Phys. 69(8-9), 2100075 (2021). https://doi.org/10.1002/prop.202100075 [arXiv:2102.11390 [hep-th]]

95. R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo, O. Hohm, Gauge invariant double copy of Yang-Mills theory: the quartic theory. Phys. Rev. D 107(12), 126015 (2023). https://doi.org/ 10.1103/PhysRevD.107.126015 [arXiv:2212.04513 [hep-th]]

96. R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo, O. Hohm, Weakly Constrained Double Field Theory as the Double Copy of Yang-Mills Theory [arXiv:2309.03289 [hep-th]]

97. K. Konosu, Y. Okawa, Nonperturbative correlation functions from homotopy algebras [arXiv:2405.10935 [hep-th]]

98. A.H. Firat, A_∞ perspective to Sen's formalism [arXiv:2405.05310 [hep-th]]

99. E. Getzler, J.D.S. Jones, A_∞ -algebras and the cyclic bar complex. Ill. J. Math. 34,256 (1990)

100. B. Zwiebach, Building string field theory around nonconformal backgrounds. Nucl. Phys. B 480, 541-572 (1996). https://doi.org/10.1016/S0550-3213(96)00502-0 [arXiv:hep-th/9606153 [hep-th]]

101. M. Markl, Loop homotopy algebras in closed string field theory. Commun. Math. Phys. 221, 367-384 (2001). https://doi.org/10.1007/PL00005575 [arXiv:hep-th/9711045 [hep-th]]

102. H. Kajiura, J. Stasheff, Homotopy algebras inspired by classical open-closed string field theory. Commun. Math. Phys. 263, 553-581 (2006). https://doi.org/10.1007/s00220-006- 1539-2 [arXiv:math/0410291 [math.QA]]

103. H. Kajiura, J. Stasheff, Homotopy algebra of open-closed strings. Geom. Topol. Monographs 13, 229-259 (2008). https://doi.org/10.2140/gtm.2008.13.229 [arXiv:hep-th/0606283 [hep-th]]

104. C. Maccaferri, A. Ruffino, J. Vošmera, The nilpotent structure of open-closed string field theory. JHEP 08, 145 (2023). https://doi.org/10.1007/JHEP08(2023)145 [arXiv:2305.02843 [hep-th]]

105. C. Maccaferri, A. Ruffino, J. Vošmera, Open-closed string field theory in the large N limit. JHEP 09, 119 (2023). https://doi.org/10.1007/JHEP09(2023)119 [arXiv:2305.02844 [hep-th]]

106. C. Maccaferri, J. Vošmera, The classical cosmological constant of open-closed string field theory. JHEP 10, 173 (2022). https://doi.org/10.1007/JHEP10(2022)173 [arXiv:2208.00410 [hep-th]]

107. H. Kajiura, Homotopy algebra morphism and geometry of classical string field theory. Nucl. Phys. B 630, 361-432 (2002). https://doi.org/10.1016/S0550-3213(02)00174-8 [arXiv:hep-th/0112228 [hep-th]]

108. A. Sen, B. Zwiebach, Background independent algebraic structures in closed string field theory. Commun. Math. Phys. 177, 305-326 (1996). https://doi.org/10.1007/BF02101895 [arXiv:hep-th/9408053 [hep-th]]

109. E. Harrelson, A.A. Voronov, J.J. Zuniga, Open-closed moduli spaces and related algebraic structures. Lett. Math. Phys. 94, 1-26 (2010). https://doi.org/10.1007/s11005-010-0418-0

110. S.F. Moosavian, Y. Zhou, On the Existence of Heterotic-String and Type-II-Superstring Field Theory Vertices [arXiv:1911.04343 [hep-th]]

111. H. Hata, M.M. Nojiri, A new symmetry in covariant open string field theory. Phys. Rev. D 36, 1193 (1987). https://doi.org/10.1103/PhysRevD.36.1193

112. A.H. Firat, String vertices for the large N limit [arXiv:2311.00747 [hep-th]]

113. T. Erler, S. Konopka, I. Sachs, One loop tadpole in heterotic string field theory. JHEP 11, 056 (2017). https://doi.org/10.1007/JHEP11(2017)056 [arXiv:1704.01210 [hep-th]]

114. A. Sen, D-instantons, string field theory and two dimensional string theory. JHEP 11, 061 (2021). https://doi.org/10.1007/JHEP11(2021)061 [arXiv:2012.11624 [hep-th]]

115. H. Erbin, S. Majumder, SL(2, C) quartic vertex for closed string field theory [arXiv:2311.07367 [hep-th]]

116. N. Moeller, Closed bosonic string field theory at quartic order. JHEP 11, 018 (2004). https://doi.org/10.1088/1126-6708/2004/11/018 [arXiv:hep-th/0408067 [hep-th]]

117. K. Strebel, Quadratic Differentials (Springer, Berlin, 1984)

118. M. Saadi, B. Zwiebach, Closed string field theory from polyhedra. Annals Phys. 192, 213 (1989). https://doi.org/10.1016/0003-4916(89)90126-7

119. T. Kugo, H. Kunitomo, K. Suehiro, Nonpolynomial closed string field theory. Phys. Lett. B 226, 48-54 (1989). https://doi.org/10.1016/0370-2693(89)90287-6

120. T. Kugo, K. Suehiro, Nonpolynomial closed string field theory: action and its gauge invariance. Nucl. Phys. B 337, 434-466 (1990). https://doi.org/10.1016/0550-3213(90)90277-K

121. N. Moeller, Closed bosonic string field theory at quintic order: five-tachyon contact term and dilaton theorem. JHEP 03, 043 (2007). https://doi.org/10.1088/1126-6708/2007/03/043 [arXiv:hep-th/0609209 [hep-th]]

122. H. Erbin, A.H. Fırat, Characterizing 4-string contact interaction using machine learning [arXiv:2211.09129 [hep-th]]

123. A.H. Firat, Bootstrapping closed string field theory. JHEP 05, 186 (2023). https://doi.org/10.1007/JHEP05(2023)186 [arXiv:2302.12843 [hep-th]]

124. B. Zwiebach, How covariant closed string theory solves a minimal area problem. Commun. Math. Phys. 136, 83-118 (1991). https://doi.org/10.1007/BF02096792

125. R. Brustein, S.P. De Alwis, Renormalization group equation and nonperturbative effects in string field theory. Nucl. Phys. B 352, 451-468 (1991). https://doi.org/10.1016/0550- 3213(91)90451-3

126. N. Moeller, H. Yang, The Nonperturbative closed string tachyon vacuum to high level. JHEP 04, 009 (2007). https://doi.org/10.1088/1126-6708/2007/04/009 [arXiv:hep-th/0609208 [hep-th]]

127. G. Zemba, B. Zwiebach, Tadpole graph in covariant closed string field theory. J. Math. Phys. 30, 2388 (1989). https://doi.org/10.1063/1.528569

128. M. Headrick, B. Zwiebach, Convex programs for minimal-area problems. Commun. Math. Phys. 377(3), 2217-2285 (2020). https://doi.org/10.1007/s00220-020-037332-1 [arXiv:1806.00449 [hep-th]]

129. M. Headrick, B. Zwiebach, Minimal-area metrics on the Swiss cross and punctured torus. Commun. Math. Phys. 377(3), 2287-2343 (2020). https://doi.org/10.1007/s00220-020- 03734-z [arXiv:1806.00450 [hep-th]]

130. U. Naseer, B. Zwiebach, Extremal isosystolic metrics with multiple bands of crossing geodesics. Adv. Theor. Math. Phys. 26(5), 1273-1346 (2022). https://doi.org/10.4310/ATMP.2022.v26.n5.a7 [arXiv:1903.11755 [math.DG]]

131. M. Gromov, Filling Riemannian manifolds. J. Diff. Geom 18, 1-147 (1983)

132. P. Buser, Geometry and Spectra of Compact Riemann Surfaces (Birkhäuser, Boston, 1992)

133. S.F. Moosavian, R. Pius, Hyperbolic geometry and closed bosonic string field theory. Part I. The string vertices via hyperbolic Riemann surfaces. JHEP 08, 157 (2019). https://doi.org/ 10.1007/JHEP08(2019)157 [arXiv:1706.07366 [hep-th]]

134. S.F. Moosavian, R. Pius, Hyperbolic geometry and closed bosonic string field theory. Part II. The rules for evaluating the quantum BV master action. JHEP 08, 177 (2019). https://doi.org/10.1007/JHEP08(2019)177 [arXiv:1708.04977 [hep-th]]

135. M. Cho, Open-closed hyperbolic string vertices. JHEP 05, 046 (2020). https://doi.org/10.1007/JHEP05(2020)046 [arXiv:1912.00030 [hep-th]]

136. A.H. Firat, Hyperbolic three-string vertex. JHEP 08, 035 (2021). https://doi.org/10.1007/ JHEP08(2021)035 [arXiv:2102.03936 [hep-th]]

137. E. Witten, Interacting field theory of open superstrings. Nucl. Phys. B 276, 291-324 (1986). https://doi.org/10.1016/05 3213(86)90298-1

138. T. Erler, S. Konopka, Vertical integration from the large hilbert space. JHEP 12, 112 (2017). https://doi.org/10.1007/JH [arXiv:1710.07232 [hep-th]]

139. T. Erler, Y. Okawa, T. Takezaki, Complete action for open superstring field theory with cyclic A_∞ structure. JHEP 08,012 (2016). https://doi.org/10.1007/JHEP08(2016)012 [arXiv:1602.02582 [hep-th]]

140. S. Konopka, I. Sachs, Open superstring field theory on the restricted hilbert space. JHEP 04, 164 (2016). https://doi.org/10.1007/JHEP04(2016)164 [arXiv:1602.02583 [hep-th]]

141. H. Kunitomo, T. Sugimoto, Heterotic string field theory with cyclic L_∞ structure. PTEP 2019(6), 063B02 (2019) [erratum: PTEP 2020 (2020) no.1, 019201]. https://doi.org/10.1093/ ptep/ptz051 [arXiv:1902.02991 [hep-th]]

142. H. Kunitomo, T. Sugimoto, Type II superstring field theory with cyclic L_∞ structure. PTEP 2020(3), 033B06 (2020). https://doi.org/10.1093/ptep/ptaa013 [arXiv:1911.04103 [hep-th]]

143. H. Kunitomo, Type II superstring field theory revisited. PTEP 2021(9), 093B03 (2021). https://doi.org/10.1093/ptep/pt [arXiv:2106.07917 [hep-th]]

144. Y. Iimori, T. Noumi, Y. Okawa, S. Torii, From the Berkovits formulation to the Witten formulation in open superstring field theory. JHEP 03, 044 (2014). https://doi.org/10.1007/ JHEP03(2014)044 [arXiv:1312.1677 [hep-th]]

145. T. Erler, Y. Okawa, T. Takezaki, A_∞ structure from the Berkovits formulation of open superstring field theory [arXiv:1505.01659 [hep-th]]

146. T. Erler, Relating Berkovits and A_∞ superstring field theories; small Hilbert space perspective. JHEP 10, 157 (2015). https://doi.org/10.1007/JHEP10(2015)157 [arXiv:1505.02069 [hep-th]]

147. T. Erler, Relating Berkovits and A_∞ superstring field theories; large Hilbert space perspective. JHEP 02, 121 (2016). https://doi.org/10.1007/JHEP02(2016)121 [arXiv:1510.00364 [hep-th]]

148. M. Kroyter, Y. Okawa, M. Schnabl, S. Torii, B. Zwiebach, Open superstring field theory I: gauge fixing, ghost structure, and propagator. JHEP 03, 030 (2012). https://doi.org/10.1007/ JHEP03(2012)030 [arXiv:1201.1761 [hep-th]]

149. K. Ohmori, Y. Okawa, Open superstring field theory based on the supermoduli space. JHEP 04, 035 (2018). https://doi.org/10.1007/JHEP04(2018)035 [arXiv:1703.08214 [hep-th]]

150. M. Kiermaier, Y. Okawa, Exact marginality in open string field theory: A General framework. JHEP 11, 041 (2009). https://doi.org/10.1088/1126-6708/2009/11/041 [arXiv:0707.4472 [hep-th]]

151. M. Kiermaier, Y. Okawa, P. Soler, Solutions from boundary condition changing operators in open string field theory. JHEP 03, 122 (2011). https://doi.org/10.1007/JHEP03(2011)122 [arXiv:1009.6185 [hep-th]]

152. L. Bonora, C. Maccaferri, D.D. Tolla, Relevant deformations in open string field theory: a simple solution for lumps. JHEP 11, 107 (2011). https://doi.org/10.1007/JHEP11(2011)107 [arXiv:1009.4158 [hep-

th]]

153. T. Erler, C. Maccaferri, String field theory solution for any open string background. JHEP 10, 029 (2014). https://doi.org/10.1007/JHEP10(2014)029 [arXiv:1406.3021 [hep-th]]

154. M. Schnabl, Phys. Lett. B 654, 194-199 (2007). https://doi.org/10.1016/j.physletb.2007.08.023 [arXiv:hep-th/0701248 [hep-th]]

155. M. Kiermaier, Y. Okawa, L. Rastelli, B. Zwiebach, Analytic solutions for marginal deformations in open string field theory. JHEP 01, 028 (2008). https://doi.org/10.1088/1126- 6708/2008/01/028 [arXiv:hep-th/0701249 [hep-th]]

156. M. Murata, M. Schnabl, Multibrane solutions in open string field theory. JHEP 07, 063 (2012). https://doi.org/10.1007/JHEP07(2012)063 [arXiv:1112.0591 [hep-th]]

157. A. Sen, String field theory as world-sheet UV regulator. JHEP 10, 119 (2019). https://doi.org/10.1007/JHEP10(2019)11 [arXiv:1902.00263 [hep-th]]

158. J. Scheinpflug, M. Schnabl, Conformal perturbation theory from open string field theory [arXiv:2301.05216 [hep-th]]

159. J. Scheinpflug, M. Schnabl, Closed string tachyon condensation revisited [arXiv:2308.16142 [hep-th]]

160. A. Sen, NonBPS states and Branes in string theory [arXiv:hep-th/9904207 [hep-th]]

161. A. Sen, Descent relations among bosonic D-branes. Int. J. Mod. Phys. A 14, 4061-4078 (1999). https://doi.org/10.1142/S0217751X99001901 [arXiv:hep-th/9902105 [hep-th]]

162. A.A. Gerasimov, S.L. Shatashvili, On exact tachyon potential in open string field theory. JHEP 10, 034 (2000). https://doi.org/10.1088/1126-6708/2000/10/034 [arXiv:hep-th/0009103 [hep-th]]

163. D. Kutasov, M. Marino, G.W. Moore, Some exact results on tachyon condensation in string field theory. JHEP 10, 045 (2000). https://doi.org/10.1088/1126-6708/2000/10/045 [arXiv:hep-th/0009148 [hep-th]]

164. D. Ghoshal, A. Sen, Normalization of the background independent open string field theory action. JHEP 11, 021 (2000). https://doi.org/10.1088/1126-6708/2000/11/021 [arXiv:hep-th/0009191 [hep-th]]

165. E. Witten, Some computations in background independent off-shell string theory. Phys. Rev. D 47, 3405-3410 (1993). https://doi.org/10.1103/PhysRevD.47.3405 [arXiv:hep-th/9210065 [hep-th]]

166. E. Witten, On background independent open string field theory. Phys. Rev. D 46, 5467-5473 (1992). https://doi.org/10.1103/PhysRevD.46.5467 [arXiv:hep-th/9208027 [hep-th]]

167. A. Sen, B. Zwiebach, Tachyon condensation in string field theory. JHEP 03, 002 (2000). https://doi.org/10.1088/1126-6708/2000/03/002 [arXiv:hep-th/9912249 [hep-th]]

168. V.A. Kostelecky, S. Samuel, The static tachyon potential in the open bosonic string theory. Phys. Lett. B 207, 169-173 (1988). https://doi.org/10.1016/0370-2693(88)91410-4

169. V.A. Kostelecky, S. Samuel, On a nonperturbative vacuum for the open bosonic string. Nucl. Phys. B 336, 263-296 (1990). https://doi.org/10.1016/0550-3213(90)90111-P

170. N. Moeller, W. Taylor, Level truncation and the tachyon in open bosonic string field theory. Nucl. Phys. B 583, 105-144 (2000). https://doi.org/10.1016/S0550-3213(00)00293- 5 [arXiv:hep-th/0002237 [hep-th]]

171. D. Gaiotto, L. Rastelli, Experimental string field theory. JHEP 08, 048 (2003). https://doi.org/10.1088/1126-6708/2003/08/048 [arXiv:hep-th/0211012 [hep-th]]

172. N. Berkovits, A. Sen, B. Zwiebach, Tachyon condensation in superstring field theory. Nucl. Phys. B 587, 147-178 (2000). https://doi.org/10.1016/S0550-3213(00)00501-0 [arXiv:hep-th/0002211 [hep-th]]

173. P.J. De Smet, J. Raeymaekers, Level four approximation to the tachyon potential in superstring field theory. JHEP 05, 051 (2000). https://doi.org/10.1088/1126-6708/2000/05/ 051 [arXiv:hep-th/0003220 [hep-

th]]

174. N. Moeller, A. Sen, B. Zwiebach, D-branes as tachyon lumps in string field theory. JHEP 08, 039 (2000). https://doi.org/10.1088/1126-6708/2000/08/039 [arXiv:hep-th/0005036 [hep-th]]

175. M. Schnabl, Analytic solution for tachyon condensation in open string field theory. Adv. Theor. Math. Phys. 10(4), 433-501 (2006). https://doi.org/10.4310/ATMP.2006.v10.n4.a1 [arXiv:hep-th/0511286 [hep-th]]

176. Y. Okawa, Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory. JHEP 04, 055 (2006). https://doi.org/10.1088/1126-6708/2006/04/ 055 [arXiv:hep-th/0603159 [hep-th]]

177. T. Erler, M. Schnabl, A simple analytic solution for tachyon condensation. JHEP 10, 066 (2009). https://doi.org/10.1088/1126-6708/2009/10/066 [arXiv:0906.0979 [hep-th]]

178. T. Erler, C. Maccaferri, String field theory solution for any open string background. Part II. JHEP 01, 021 (2020). https://doi.org/10.1007/JHEP01(2020)021 [arXiv:1909.11675 [hep-th]]

179. L. Rastelli, B. Zwiebach, Tachyon potentials, star products and universality. JHEP 09, 038 (2001). https://doi.org/10.1088/1126-6708/2001/09/038 [arXiv:hep-th/0006240 [hep-th]]

180. L. Rastelli, A. Sen, B. Zwiebach, Half strings, projectors, and multiple D-branes in vacuum string field theory. JHEP 11, 035 (2001). https://doi.org/10.1088/1126-6708/2001/11/035 [arXiv:hep-th/0105058 [hep-th]]

181. D.J. Gross, W. Taylor, Split string field theory. 1. JHEP 08, 009 (2001). https://doi.org/10.1088/1126-6708/2001/08/009 [arXiv:hep-th/0105059 [hep-th]]

182. L. Rastelli, A. Sen, B. Zwiebach, Boundary CFT construction of D-branes in vacuum string field theory. JHEP 11, 045 (2001). https://doi.org/10.1088/1126-6708/2001/11/045 [arXiv:hep-th/0105168 [hep-th]]

183. L. Rastelli, A. Sen, B. Zwiebach, Vacuum string field theory [arXiv:hep-th/0106010 [hep-th]]

184. D. Gaiotto, L. Rastelli, A. Sen, B. Zwiebach, Star algebra projectors. JHEP 04, 060 (2002). https://doi.org/10.1088/1126-6708/2002/04/060 [arXiv:hep-th/0202151 [hep-th]]

185. S. Zeze, Regularization of identity based solution in string field theory. JHEP 10, 070 (2010). https://doi.org/10.1007/JHEP10(2010)070 [arXiv:1008.1104 [hep-th]]

186. E.A. Arroyo, Comments on regularization of identity based solutions in string field theory. JHEP 11, 135 (2010). https://doi.org/10.1007/JHEP11(2010)135 [arXiv:1009.0198 [hep-th]]

187. M. Schnabl, Algebraic solutions in open string field theory—a lightning review. Acta Polytech. 50(3), 102-108 (2010). https://doi.org/10.14311/1213 [arXiv:1004.4858 [hep-th]]

188. T. Erler, Analytic solution for tachyon condensation in Berkovits' open superstring field theory. JHEP 11 (2013), 007 https://doi.org/10.1007/JHEP11(2013)007 [arXiv:1308.4400 [hep-th]]

189. T. Erler, Marginal solutions for the superstring. JHEP 07, 050 (2007). https://doi.org/10.1088/1126-6708/2007/07/050 [arXiv:0704.0930 [hep-th]]

190. Y. Okawa, Analytic solutions for marginal deformations in open superstring field theory. JHEP 09, 084 (2007). https://doi.org/10.1088/1126-6708/2007/09/084 [arXiv:0704.0936 [hep-th]]

191. Y. Okawa, Real analytic solutions for marginal deformations in open superstring field theory. JHEP 09, 082 (2007). https://doi.org/10.1088/1126-6708/2007/09/082 [arXiv:0704.3612 [hep-th]]

192. N. Moeller, B. Zwiebach, Dynamics with infinitely many time derivatives and rolling tachyons. JHEP 10, 034 (2002). https://doi.org/10.1088/1126-6708/2002/10/034 [arXiv:hep-th/0207107 [hep-th]]

193. S. Hellerman, M. Schnabl, Light-like tachyon condensation in Open String Field Theory. JHEP 04, 005 (2013). https://doi.org/10.1007/JHEP04(2013)005 [arXiv:0803.1184 [hep-th]]

194. J.A. Shapiro, C.B. Thorn, Closed string-open string transitions and witten's string field theory. Phys. Lett. B 194, 43-48 (1987). https://doi.org/10.1016/0370-2693(87)90766-0

195. D.Z. Freedman, S.B. Giddings, J.A. Shapiro, C.B. Thorn, The nonplanar one loop amplitude in witten's string field theory. Nucl. Phys. B 298, 253 (1988). https://doi.org/10.1016/0550- 3213(88)90268-4

196. A. Hashimoto, N. Itzhaki, Observables of string field theory. JHEP 01, 028 (2002). https:// doi.org/10.1088/1126-6708/2002/01/028 [arXiv:hep-th/0111092 [hep-th]]

197. D. Gaiotto, L. Rastelli, A. Sen, B. Zwiebach, Ghost structure and closed strings in vacuum string field theory. Adv. Theor. Math. Phys. 6, 403-456 (2003), https://doi.org/10.4310/ ATMP.2002.v6.n3.a1 [arXiv:hep-th/0111129 [hep-th]]

198. I. Ellwood, The Closed string tadpole in open string field theory. JHEP 08, 063 (2008). https://doi.org/10.1088/1126-6708/2008/08/063 [arXiv:0804.1131 [hep-th]]

199. M. Kudrna, C. Maccaferri, M. Schnabl, Boundary state from ellwood invariants. JHEP 07, 033 (2013). https://doi.org/10.1007/JHEP07(2013)033 [arXiv:1207.4785 [hep-th]]

200. M. Kiermaier, Y. Okawa, B. Zwiebach, The boundary state from open string fields [arXiv:0810.1737 [hep-th]]

201. E. Witten, Superstring Perturbation Theory Revisited [arXiv:1209.5461 [hep-th]]

202. R. Pius, A. Rudra, A. Sen, Mass renormalization in string theory: special states. JHEP 07, 058 (2014). https://doi.org/10.1007/JHEP07(2014)058 [arXiv:1311.1257 [hep-th]]

203. R. Pius, A. Rudra, A. Sen, Mass renormalization in string theory: general states. JHEP 07, 062 (2014). https://doi.org/10.1007/JHEP07(2014)062 [arXiv:1401.7014 [hep-th]]

204. A. Sen, Gauge invariant 1PI effective superstring field theory: inclusion of the ramond sector. JHEP 08, 025 (2015). https://doi.org/10.1007/JHEP08(2015)025 [arXiv:1501.00988 [hep-th]]

205. W. Fischler, L. Susskind, Dilaton tadpoles, string condensates and scale invariance. Phys. Lett. B 171, 383-389 (1986). https://doi.org/10.1016/0370-2693(86)91425-5

206. R. Pius, A. Rudra, A. Sen, String perturbation theory around dynamically shifted vacuum. JHEP 10, 070 (2014). https://doi.org/10.1007/JHEP10(2014)070 [arXiv:1404.6254 [hep-th]]

207. A. Sen, Supersymmetry restoration in superstring perturbation theory. JHEP 12, 075 (2015). https://doi.org/10.1007/J [arXiv:1508.02481 [hep-th]]

208. M. Cho, S. Collier, X. Yin, Strings in Ramond-Ramond Backgrounds from the Neveu-Schwarz-Ramond Formalism. JHEP 12, 123 (2020). https://doi.org/10.1007/ JHEP12(2020)123 [arXiv:1811.00032 [hep-th]]

209. M. Cho, M. Kim, A Worldsheet Description of Flux Compactifications [arXiv:2311.04959 [hep-th]]

210. J. Polchinski, String theory, in An Introduction to the Bosonic String, vol. 1 (Cambridge University, Cambridge, 2007). ISBN 978-0-511-25227-3, 978-0-521-67227-6, 978-0-521- 63303-1. https://doi.org/10.1017/CBO97805118160

211. J. Polchinski, String theory. Superstring Theory and Beyond, vol. 2 (Cambridge University, Cambridge, 2007). ISBN 978-0-511-25228-0, 978-0-521-63304-8, 978-0-521-67228-3. https://doi.org/10.1017/CBO9780511618123

212. A. Sen, Normalization of type IIB D-instanton amplitudes. JHEP 12, 146 (2021). https://doi.org/10.1007/JHEP12(2021 [arXiv:2104.11109 [hep-th]]

213. A. Sen, Muti-instanton amplitudes in type IIB string theory. JHEP 12, 065 (2021). https:// doi.org/10.1007/JHEP12(20 [arXiv:2104.15110 [hep-th]]

214. S. Alexandrov, A. Sen, B. Stefański, D-instantons in Type IIA string theory on Calabi-Yau threefolds. JHEP 11, 018 (2021). https://doi.org/10.1007/JHEP11(2021)018 [arXiv:2108.04265 [hep-th]]

215. S. Alexandrov, A. Sen, B. Stefański, Euclidean D-branes in type IIB string theory on Calabi-Yau threefolds. JHEP 12, 044 (2021). https://doi.org/10.1007/JHEP12(2021)044 [arXiv:2110.06949 [hep-th]]

216. S. Alexandrov, A.H. Firat, M. Kim, A. Sen, B. Stefański, D-instanton induced superpotential. JHEP 07, 090 (2022). https://doi.org/10.1007/JHEP07(2022)090 [arXiv:2204.02981 [hep-th]]

217. A. Sen, Normalization of D-instanton amplitudes. JHEP 11, 077 (2021). https://doi.org/10.1007/JHEP11(2021)077 [arXiv:2101.08566 [hep-th]]

218. M.B. Green, M. Gutperle, Effects of D instantons. Nucl. Phys. B 498, 195-227 (1997). https://doi.org/10.1016/S0550-3213(97)00269-1 [arXiv:hep-th/9701093 [hep-th]]

219. N.B. Agmon, B. Balthazar, M. Cho, V.A. Rodriguez, X. Yin, D-instanton Effects in Type IIB String Theory [arXiv:2205.00609 [hep-th]]

220. D.S. Eniceicu, R. Mahajan, C. Murdia, A. Sen, Normalization of ZZ instanton amplitudes in minimal string theory. JHEP 07, 139 (2022). https://doi.org/10.1007/JHEP07(2022)139 [arXiv:2202.03448 [hep-th]]

221. D.S. Eniceicu, R. Mahajan, C. Murdia, A. Sen, Multi-instantons in minimal string theory and in matrix integrals. JHEP 10, 065 (2022). https://doi.org/10.1007/JHEP10(2022)065 [arXiv:2206.13531 [hep-th]]

222. J. Chakravarty, A. Sen, Normalization of D instanton amplitudes in two dimensional type 0B string theory. JHEP 02, 170 (2023). https://doi.org/10.1007/JHEP02(2023)170 [arXiv:2207.07138 [hep-th]]

223. S. Alexandrov, R. Mahajan, A. Sen, Instantons in sine-Liouville theory [arXiv:2311.04969 [hep-th]]

224. A. Sen, Fixing an ambiguity in two dimensional string theory using string field theory. JHEP 03, 005 (2020). https://doi.org/10.1007/JHEP03(2020)005 [arXiv:1908.02782 [hep-th]]

225. A. Sen, D-instanton perturbation theory. JHEP 08, 075 (2020). https://doi.org/10.1007/ JHEP08(2020)075 [arXiv:2002.04043 [hep-th]]

226. D.S. Eniceicu, R. Mahajan, P. Maity, C. Murdia, A. Sen, The ZZ annulus one-point function in non-critical string theory: a string field theory analysis. JHEP 12, 151 (2022). https://doi.org/10.1007/JHEP12(2022)151 [arXiv:2210.11473 [hep-th]]

227. A. Sen, Cutkosky rules and unitarity (violation) in D-instanton amplitudes. JHEP 07, 205 (2021). https://doi.org/10.1007/JHEP07(2021)205 [arXiv:2012.00041 [hep-th]]

228. A. Sen, Unitarity of superstring field theory. JHEP 12, 115 (2016). https://doi.org/10.1007/ JHEP12(2016)115 [arXiv:1607.08244 [hep-th]]

229. R. Pius, A. Sen, Cutkosky rules for superstring field theory. JHEP 10, 024 (2016). [erratum: JHEP 09, 122 (2018)]. https://doi.org/10.1007/JHEP10(2016)024 [arXiv:1604.01783 [hep-th]]

230. M.J.G. Veltman, Unitarity and causality in a renormalizable field theory with unstable particles. Physica 29, 186-207 (1963). https://doi.org/10.1016/S0031-8914(63)80277-3

231. A. Sen, Equivalence of two contour prescriptions in superstring perturbation theory. JHEP 04, 025 (2017). https://doi.org/10.1007/JHEP04(2017)025 [arXiv:1610.00443 [hep-th]]

232. B. Balthazar, V.A. Rodriguez, X. Yin, ZZ instantons and the non-perturbative dual of $c = 1$ string theory. JHEP 05, 048 (2023). https://doi.org/10.1007/JHEP05(2023)048 [arXiv:1907.07688 [hep-th]]

233. J. Bros, H. Epstein, V.J. Glaser, Some rigorous analyticity properties of the four-point function in momentum space. Nuovo Cim. 31, 1265-1302 (1964). https://doi.org/10.1007/ BF02733596

234. J. Bros, H. Epstein, V. Glaser, A proof of the crossing property for two-particle amplitudes in general quantum field theory. Commun. Math. Phys. 1(3), 240-264 (1965). https://doi.org/ 10.1007/BF01646307

235. C. De Lacroix, H. Erbin, A. Sen, Analyticity and crossing symmetry of superstring loop amplitudes. JHEP 05, 139 (2019). https://doi.org/10.1007/JHEP05(2019)139 [arXiv:1810.07197 [hep-th]]

236. A. Belopolsky, New geometrical approach to superstrings [arXiv:hep-th/9703183 [hep-th]]

237. A. Belopolsky, Picture changing operators in supergeometry and superstring theory [arXiv:hep-th/9706033 [hep-th]]

238. C. Wang, X. Yin, On the equivalence between SRS and PCO formulations of superstring perturbation theory. JHEP 03, 139 (2023). https://doi.org/10.1007/JHEP03(2023)139 [arXiv:2205.01106 [hep-th]]

239. C. Wang, X. Yin, Supermoduli and PCOs at Genus Two. https://doi.org/10.1007/ JHEP01(2023)144 [arXiv:2205.10377 [hep-th]]

240. T. Takezaki, Open superstring field theory including the Ramond sector based on the supermoduli space [arXiv:1901.02176 [hep-th]]

241. B. Zwiebach, New moduli spaces from string background independence consistency conditions. Nucl. Phys. B 480, 507-540 (1996). https://doi.org/10.1016/S0550- 3213(96)00501-9 [arXiv:hep-th/9605075 [hep-th]]

242. L. Rastelli, A. Sen, B. Zwiebach, String field theory around the tachyon vacuum. Adv. Theor. Math. Phys. 5, 353-392 (2002). https://doi.org/10.4310/ATMP.2001.v5.n2.a5 [arXiv:hep-th/0012251 [hep-th]]

243. L. Rastelli, A. Sen, B. Zwiebach, Classical solutions in string field theory around the tachyon vacuum. Adv. Theor. Math. Phys. 5, 393-428 (2002). https://doi.org/10.4310/ATMP.2001.v5.n2.a6 [arXiv:hep-th/0102112 [hep-th]]

244. Y. Okawa, Open string states and D-brane tension from vacuum string field theory. JHEP 07, 003 (2002). https://doi.org/10.1088/1126-6708/2002/07/003 [arXiv:hep-th/0204012 [hep-th]]

245. N. Drukker, Y. Okawa, Vacuum string field theory without matter-ghost factorization. JHEP 06, 032 (2005). https://doi.org/10.1088/1126-6708/2005/06/032 [arXiv:hep-th/0503068 [hep-th]]

246. M. Schnabl, G. Stettinger, Open string field theory with stubs. JHEP 07, 032 (2023). https://doi.org/10.1007/JHEP07(2023)032 [arXiv:2301.13182 [hep-th]]

247. M. Schnabl, G. Stettinger, More on stubs in open string field theory [arXiv:2402.00308 [hep-th]]

248. H. Erbin, A.H. Fırat, Open string stub as an auxiliary string field [arXiv:2308.08587 [hep-th]]

249. C. Maccaferri, R. Poletti, A. Ruffino, B. Valsesia, Adding stubs to quantum string field theories [arXiv:2403.10471 [hep-th]]

250. C. Chiaffrino, I. Sachs, QFT with stubs. JHEP 06, 120 (2022). https://doi.org/10.1007/ JHEP06(2022)120 [arXiv:2108.04312 [hep-th]]

251. T. Erler, A.H. Firat, Wilsonian effective potentials and closed string field theory. JHEP 02, 018 (2024). https://doi.org/10.1007/JHEP02(2024)018 [arXiv:2311.17322 [hep-th]]

252. H. Erbin, M. Médevielle, Closed string theory without level-matching at the free level. JHEP 03, 091 (2023). https://doi.org/10.1007/JHEP03(2023)091 [arXiv:2209.05585 [hep-th]]

253. Y. Okawa, R. Sakaguchi, Closed string field theory without the level-matching condition [arXiv:2209.06173 [hep-th]]

254. R.M. Wald, Black hole entropy is the Noether charge. Phys. Rev. D 48(8), R3427-R3431 (1993). https://doi.org/10.1103/PhysRevD.48.R3427 [arXiv:gr-qc/9307038 [gr-qc]]

255. G.W. Gibbons, S.W. Hawking, Action integrals and partition functions in quantum gravity. Phys. Rev. D 15, 2752-2756 (1977). https://doi.org/10.1103/PhysRevD.15.2752

256. T. Erler, The closed string field theory action vanishes. JHEP 10, 055 (2022). https://doi.org/10.1007/JHEP10(2022)055 [arXiv:2204.12863 [hep-th]]

257. A. Ahmadain, A.C. Wall, Off-Shell Strings I: S-matrix and Action [arXiv:2211.08607 [hep-th]]

258. A. Ahmadain, A.C. Wall, Off-Shell Strings II: Black Hole Entropy [arXiv:2211.16448 [hep-th]]

259. P. Kraus, A. Ryzhov, M. Shigemori, Strings in noncompact space-times: boundary terms and conserved charges. Phys. Rev. D 66, 106001 (2002). https://doi.org/10.1103/PhysRevD.66.106001 [arXiv:hep-th/0206080 [hep-th]]

260. S. Kim, P. Kraus, R.M. Myers, Systematics of boundary actions in gauge theory and gravity. JHEP 04, 121 (2023). https://doi.org/10.1007/JHEP04(2023)121 [arXiv:2301.02964 [hep-th]]

261. Y. Okawa, Nonperturbative definition of closed string theory via open string field theory [arXiv:2006.16449 [hep-th]]

262. C. Chiaffrino, T. Ersoy, O. Hohm, Holography as Homotopy [arXiv:2307.08094 [hep-th]]

263. R. Gopakumar, C. Vafa, On the gauge theory/geometry correspondence. Adv. Theor. Math. Phys. 3, 1415-1443 (1999). https://doi.org/10.4310/ATMP.1999.v3.n5.a5 [arXiv:hep-th/9811131 [hep-th]]

264. D. Gaiotto, L. Rastelli, A Paradigm of open/closed duality: liouville D-branes and the Kontsevich model. JHEP 07, 053 (2005). https://doi.org/10.1088/1126-6708/2005/07/053 [arXiv:hep-th/0312196 [hep-th]]

265. J.M. Maldacena, H. Ooguri, Strings in AdS(3) and SL(2, R) WZW model 1.: the Spectrum. J. Math. Phys. 42, 2929-2960 (2001). https://doi.org/10.1063/1.1377273 [arXiv:hep-th/0001053 [hep-th]]

266. J.M. Maldacena, H. Ooguri, J. Son, Strings in AdS(3) and the SL(2, R) WZW model. Part 2. Euclidean black hole. J. Math. Phys. 42, 2961-2977 (2001). https://doi.org/10.1063/1.1377039 [arXiv:hep-th/0005183 [hep-th]]

267. J.M. Maldacena, H. Ooguri, Strings in AdS(3) and the SL(2, R) WZW model. Part 3. Correlation functions. Phys. Rev. D 65, 106006 (2002). https://doi.org/10.1103/PhysRevD.65.106006 [arXiv:hep-th/0111180 [hep-th]]

268. M.R. Gaberdiel, R. Gopakumar, Tensionless string spectra on AdS3. JHEP 05, 085 (2018). https://doi.org/10.1007/JHE [arXiv:1803.04423 [hep-th]]

269. L. Eberhardt, M.R. Gaberdiel, R. Gopakumar, The worldsheet dual of the symmetric product CFT. JHEP 04, 103 (2019). https://doi.org/10.1007/JHEP04(2019)103 [arXiv:1812.01007 [hep-th]]

270. L. Eberhardt, M.R. Gaberdiel, R. Gopakumar, Deriving the AdS_3/CFT_2 correspondence. JHEP 02, 136 (2020). https://doi.org/10.1007/JHEP02(2020)136 [arXiv:1911.00378 [hep-th]]